Geometric Calculus 1, 201.1.1031

Homework 6

Fall 2019 (D.Kerner)

(1) (a) Let $S^1 \times S^1 = \{\underline{x} |\ x_1^2 + x_2^2 = 1 = x_3^2 + x_4^2\} \subset \mathbb{R}^4$. Find the equations for the tangent plane to this subset at a point $\underline{x}_0$. (e.g. for $x_{2,0}, x_{4,0} > 0$ present $S^1 \times S^1$ as the graph of a function.) Check that none of these planes intersects the subset $\Big(Ball_1(0,0) \times \mathbb{R}^2\Big) \cup \Big(\mathbb{R}^2 \times Ball_1(0,0)\Big)$.

(b) Let $\mathbb{R}^n \supseteq \mathscr{D}_f \xrightarrow{f} \mathbb{R}^1$ be differentiable. Fix a plane, $L \subset \mathbb{R}^n$. Verify: $T\Gamma_{f|_L} = T\Gamma_f \cap \{L \times \mathbb{R}^1\}$.

(c) Take a differentiable function $\mathbb{R}^n \xrightarrow{f} \mathbb{R}$ and a point $(\underline{x}, x_{n+1}) \in \mathbb{R}^{n+1}$, $f(\underline{x}) \neq x_{n+1}$. Prove:

(i) There exists a point $\underline{\tilde{x}} \in \mathbb{R}^n$ such that $d\big((\underline{x}, x_{n+1}), \Gamma_f\big) = d\big((\underline{x}, x_{n+1}), (\underline{\tilde{x}}, f(\underline{\tilde{x}}))\big)$.

(ii) The line $\overline{(\underline{x}, x_{n+1}), (\underline{\tilde{x}}, f(\underline{\tilde{x}}))}$ is orthogonal to the tangent plane to Γ_f at $(\underline{\tilde{x}}, f(\underline{\tilde{x}}))$.

(2) (a) $\mathbb{R}^2 \supseteq \mathscr{D}_f \xrightarrow{f} \mathbb{R}$ is called independent of x_2 if $f(x_1, x_2) = g(x_1)$, for some function g of one variable. Prove: if f is C^1, and $\partial_{x_2} f = 0$ on a convex $\mathscr{D}_f$, then f is independent of x_2. Show by an example that the convexity of $\mathscr{D}_f$ cannot be weakened to path-connectedness.

(b) Let $\mathbb{R}^2 \supseteq \mathscr{D}_f \xrightarrow{f} \mathbb{R}$ be differentiable. Fix pairwise distinct points, $p_1, p_2, p_3 \in \mathscr{D}_f$. Let L be the (unique) plane through the points $(p_1, f(p_1))$, $(p_2, f(p_2))$, $(p_3, f(p_3))$. (Dis)prove: there exists a plane parallel to L and tangent to Γ_f.

(c) Prove: if f is differentiable and $f(0) = 0$ then $f(\underline{x}) = \sum\limits_{i=1}^{n} x_i g_i(\underline{x})$, for some continuous functions $\{g_i\}$. (Hint: define $h_{\underline{x}}(t) = f(t \cdot \underline{x})$ and note $f(x) = \int\limits_0^1 h'_{\underline{x}}(t)dt$.)

(3) A function $\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_r} \xrightarrow{\phi} \mathbb{R}^m$ is called multi-linear if it is linear in each component (i.e. $\phi(\underline{a}^{(1)}, \dots, \underline{a}^{(j-1)}, \underline{x}^{(j)}, \underline{a}^{(j+1)}, \dots, \underline{a}^{(r)})$ is linear in $\underline{x}^{(j)}$, for each j.)

(a) Check that for $r = 2$, $m = 1$ such a function can be presented as $\underline{x}^{(1)} \cdot A \cdot (\underline{x}^{(2)})^t$, for some $A \in Mat_{n_1 \times n_2}(\mathbb{R})$. Generalize this to $r \geq 2$, $m = 1$.

(b) Denote the set of multilinear functions by $Mul^r(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_r}, \mathbb{R}^m)$. Prove: this is an $\mathbb{R}$-vector space. Construct the isomorphism of vector spaces: $Mul^r(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_r}, \mathbb{R}^m) \xrightarrow{\sim} Hom(\mathbb{R}^{n_1}, Hom(\mathbb{R}^{n_2}, \dots Hom(\mathbb{R}^{n_r}, \mathbb{R}^m)_{\dots})$. (Therefore r'th derivative of a function $\mathbb{R}^n \xrightarrow{f} \mathbb{R}^1$ is an element of $Mul^r(\mathbb{R}^n \times \cdots \times \mathbb{R}^n, \mathbb{R}^m)$.)

(c) For any $\phi \in Mul^r(\prod^r \mathbb{R}^n, \mathbb{R}^m)$ prove: $\lim\limits_{v \to 0} \frac{\phi(v, \dots, v)}{|v|^{r-1}} = 0$.

(d) Compute $\phi'|_0$, $\phi''|_0$, $\dots$, $\phi^{(r-1)}|_0$. Compute $\phi'|_{(\underline{x}^{(1)}, \dots, \underline{x}^{(r)})}(\vec{v}^{(1)}, \dots, \vec{v}^{(r)})$.

(e) Prove: $\phi^{(r+1)} = 0$ (at any point). In particular, any multi-linear function is C^∞.

(4) (a) Check whether $\partial^2_{xy} f|_{(0,0)} = \partial^2_{yx} f|_{(0,0)}$ holds for $f(x,y) = \begin{cases} xy\frac{x^2 - y^2}{x^2 + y^2} : & (x,y) \neq (0,0) \\ 0 : & (x,y) = (0,0) \end{cases}$.

(b) We have proved in the class: $\partial^2_{x_i x_j} f = \partial^2_{x_j x_i} f$ for C^2-functions. Prove: for C^k-functions the derivatives up to order k do not depend on the order of differentiation. (Hint: use the C^2-case.)

(c) Expand $arctan(\frac{x+y}{1+xy})$ into Taylor series up to order 3 at $(0,0)$.

(d) Prove: the order-k Taylor polynomial of a C^k-function is unique. Namely, if $\lim\limits_{|\underline{x}| \to 0} \frac{f(\underline{x}) - P(\underline{x})}{|\underline{x}|^k} = 0$ for a polynomial P of degree $\leq k$, then P is unique.