Geometric Calculus 1, 201.1.1031

Homework 7

Fall 2019 (D.Kerner)

(1) (a) Denote by $Taylor_{\underline{x}_0}[f(\underline{x})]$ the Taylor series of a C^∞ function f at a point $\underline{x}_0$.

(i) Verify: $\sum a_{\underline{m}} \underline{x}^{\underline{m}}$ is the Taylor series of f iff $f(\underline{x}) - \sum a_{\underline{m}} \underline{x}^{\underline{m}} \in o(||\underline{x} - \underline{x}_0||^p)$ for any $p \in \mathbb{N}$.

(ii) Relate $Taylor_{\underline{x}_0}[f(\underline{x}) \cdot g(\underline{x})]$ to $Taylor_{\underline{x}_0}[f(\underline{x})]$, $Taylor_{\underline{x}_0}[g(\underline{x})]$. (No need to write the series in the explicit form.)

(iii) Relate $Taylor_{\underline{x}_0}[f(g(\underline{x}))]$ to $Taylor_{g(\underline{x}_0)}[f(\underline{y})]$, $Taylor_{\underline{x}_0}[g(\underline{x})]$.

(iv) Expand $f(x,y) = tan(e^{x^2-y^3} - e^{y^3-x^2})$ up to order 5, i.e. up to $O(x^6, x^5y, \dots, y^6)$.

(b) Verify: $f'|_{\underline{x}_0}(\vec{v}) = (\sum v_i \partial_i) f|_{\underline{x}_0}$. By repeated differentiation verify: $f^{(k)}|_{\underline{x}_0}(\vec{v}, \vec{v}, \dots, \vec{v}) = (\sum v_i \partial_i)^k f|_{\underline{x}_0}$.

(c) Suppose all the derivatives of a C^∞ function f on $Ball_r(\underline{x}_0)$ are bounded: $|f^{(k)}|_c(\vec{v}, \vec{v}, \dots, \vec{v})| \le \frac{C \cdot k!}{r^k} \cdot ||\vec{v}||^k$, for some $C \in \mathbb{R}$ and any $c \in Ball_r(\underline{x}_0)$, $\vec{v} \in \mathbb{R}^n$. Prove: the series $Taylor_{\underline{x}_0}[f]$ converges absolutely to the function f on $Ball_r(\underline{x}_0)$. (A function that coincides with its Taylor series is called "real analytic".)

(2) (a) Find and classify the extrema of the following functions: i. $f(x,y) = e^{sin(x)sin(y)}$ ii. $f(x,y,z) = xy + yz + xz$ iii. $f(\underline{x}) = \underline{x} \cdot A \cdot \underline{x}^t$, here $A = A^t \in Mat_{n \times n}(\mathbb{R})$.

(b) Prove Rolle's theorem: if $\mathscr{D}_f \xrightarrow{f} \mathbb{R}^n$ is continuous on a compact set $\mathscr{D}_f$, is differentiable in the interior $Int(\mathscr{D}_f)$, and satisfies $f|_{\partial \mathscr{D}_f} = const$, then $f'|_c = 0$ for some $c \in Int(\mathscr{D}_f)$.

(c) Solve question 1.c.ii from homework 6 using the necessary criterion for critical points.

(d) Given the points $(x_1, y_1), \dots, (x_N, y_N) \in \mathbb{R}^2$ define the "total error", $\sigma(a,b) := \sum (y_i - ax_i - b)^2$. Prove: σ achieves its absolute minimum for (a,b) satisfying: $\begin{bmatrix} \sum x_i^2 & \sum x_i \\ \sum x_i & N \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum x_i y_i \\ \sum y_i \end{bmatrix}$. (wiki: Least Squares)

(3) (Dis)prove the following statements. (The hints are on the next page.)

(a) If $\mathbb{R}^2 \xrightarrow{f} \mathbb{R}$ has a unique critical point, which is a local minimum, then f is bounded from below.

(b) If $\mathbb{R}^2 \xrightarrow{f} \mathbb{R}$ has an infinite number of local maxima then it has at least one local minimum.

(c) Given a polynomial $\mathbb{R}^2 \xrightarrow{p} \mathbb{R}$, whose restriction to each line through the origin, $p|_L$, has a local minimum at 0. Then p has a local minimum at $(0,0)$.

(4) (a) We have defined the notion of min/max/saddle of $\mathscr{D}_f \xrightarrow{f} \mathbb{R}$ without fixing the coordinates. Check that the conditions "$grad(f)|_{\underline{x}_0} = 0$, $f''|_{\underline{x}_0}$ is positive/negative definite" are preserved under C^2 local coordinate changes.

(b) Find and classify the critical points of $f(x,y) = ln(arctan\frac{y}{x}) \cdot sin(\sqrt{x^2+y^2})$.

(c) Identify the space of symmetric matrices, $Mat^{sym}_{n \times n}(\mathbb{R})$, with $\mathbb{R}^{\binom{n+1}{2}}$. The ordinary norm of $\mathbb{R}^{\binom{n+1}{2}}$ induces the norm on $Mat^{sym}_{n \times n}(\mathbb{R})$. Prove: the subsets of positive/negative definite matrices are open. Thus being positive/negative definite is stable under small perturbations.

(5) (a) Fix some points $\underline{x}^{(1)}, \underline{x}^{(2)}, \underline{x}^{(3)}$ in $\mathbb{R}^n$. Take the sum of distances to a given point, $f(\underline{x}) = \sum_{i=1}^{3} ||\underline{x} - \underline{x}^{(i)}||$. (Dis)Prove: the absolute minimum of this function is obtained at a point where the angles between the lines $\{\overline{\underline{x}, \underline{x}^{(i)}}\}_i$ are all 120°.

(b) Define the map $\mathbb{R}^2_{\phi,\psi} \xrightarrow{f} \mathbb{R}^3_{xyz}$ by $f_x(\phi,\theta) = (R + rsin(\phi))cos(\theta)$, $f_y(\phi,\theta) = (R + rsin(\phi))sin(\theta)$, $f_z(\phi,\theta) = rcos(\phi)$.

(i) Find and classify the critical points of f_x.

(ii) Identify the image $\underline{f}(\mathbb{R}^2) \subset \mathbb{R}^3$. (Seen hwk.0) Explain geometrically the results of (a).

Hints to question 3:
$f(x,y) = x^2 + y^2(1-x)^3$ $\qquad$ $f(x,y) = (1+e^y)cos(x) - ye^y$ $\qquad$ $f(x,y) = (y-x^2)(y-3x^2)$.