Geometric Calculus 1, 201.1.1031

Homework 8

Fall 2019 (D.Kerner)

The hints to some questions are at the next page.

(1) (a) Define $x(t) = ln(t) + sin(t)$, $y(t) = arctan(t) - e^t$. Compute $\frac{d^2y}{dx^2}|_{t=1}$.
(b) $z(x, y)$ is C^1, defined by $e^z + zy + xy = 1$, with $z(0, 0) = 0$. Find/classify its critical points.
(c) Find/classify the critical points of $y(x)$, defined by $ln\sqrt{x^2 + y^2} = arctan\frac{y}{x}$, for $y \neq x$.
(d) Suppose $\mathbb{R}^n \supseteq \mathscr{D}_f \xrightarrow{f} \mathbb{R}^n$ is C^1. Suppose the equation $f(\underline{x}) = \underline{y}_0$ has k solutions, $\underline{x}^{(1)}, \ldots, \underline{x}^{(k)}$ and they satisfy $\{det[f'|_{\underline{x}^{(i)}}] \neq 0\}_i$. Prove: for some $\delta > 0$ and any $\underline{y} \in Ball_\delta(\underline{y}_0)$ the equation $f(\underline{x}) = \underline{y}$ has at least k solutions.
(e) Let $f(\underline{x}) = 0$ be a C^∞-equation, and assume for each solving point, $f(\underline{x}_0) = 0$, holds: $f'|_{\underline{x}_0} = 0$. Does this imply that the equation cannot be resolved by a C^1-function?
(f) Can the C^1-assumption of Implicit Function Theorem be weakened to "$F(\underline{x}, y)$ is differentable"?

(2) (a) Find the tangent plane to the standard sphere $S^{n-1} \subset \mathbb{R}^n$.
(b) Prove: $grad(f)$ is orthogonal to the level curve $\{f(x, y) = c\} \subset \mathbb{R}^2$, for f - C^1, at all the smooth points of the curve. (Where $grad(f)|_p \neq 0$.)
(c) Suppose a level curve of $f(x, y)$ is defined by $y^2 = x^2$. Prove: $grad(f)|_{0,0} = 0$.
(Note: this does not imply $f(x, y) = y^2 - x^2$, e.g. all $f(x) = x^n$ have the same zero level set)
(d) In the following cases find the (non)smooth points:
i. $\{(x, y, z)|\ (x^2 + y^2 + z^2)^{17} = \frac{z^3}{(x^2+y^2)^8}\}$ ii. $\{\underline{x}|\ ||L(\underline{x})|| = 1\} \subset \mathbb{R}^n$ for $L \in Hom(\mathbb{R}^n, \mathbb{R}^m)$.
iii. $\{(x, y)|\ y^2 = \prod_{i=1}^{d}(x - \alpha_i)^2\}$. Sketch this curve in small neighborhoods of non-smooth points.
(e) Verify: the definition of smoothness of the hypersurface at a point $p \in \{f(\underline{x}) = 0\}$ does not depend on the choice of the local coordinates.

(3) The angle of intersection of two hypersurfaces, $X_f = \{f(\underline{x}) = 0\}$, $X_g = \{g(\underline{x}) = 0\}$, at a smooth point $p \in X_f \cap X_g \subset \mathbb{R}^n$, is the angle between the coresponding normals.
(a) Express the angle of intersection of $\partial Ball_r(\underline{a})$, $\partial Ball_R(\underline{b})$ at $\underline{p}$ via the vectors $\underline{a} - \underline{p}$, $\underline{b} - \underline{p}$.
(b) Express the angle of intersection of X_f, X_g at p via $f'|_p$, $g'|_p$.
(c) Verify: the angle of intersection of X_f, X_g is preserved under shifts/orthogonal transformations.
(d) Let $f(\underline{x}) = \sum x_i^{2a}$ and $g(\underline{x}) = f(\underline{x}) + p_{<2a}(\underline{x})$, where $a \in \mathbb{N}$ and $p_{<2a}$ is a polynomial of degree $< 2a$. Prove: "the graphs are tangent at ∞". Namely: if $\mathcal{N}_{(f,\underline{x})}$, $\mathcal{N}_{(g,\underline{x})}$ are the normals to the graphs at a points $\underline{x} \in \Gamma_f \cap \Gamma_g$, then the angle between $\mathcal{N}_{(f,\underline{x})}$, $\mathcal{N}_{(g,\underline{x})}$ goes to 0 as $|\underline{x}| \to \infty$.

(4) (a) Let $f(t)$ be C^1. Suppose $z(x, y)$ satisfies the equations $ln(z) + x \cdot cos(t) + y \cdot sin(t) = f(t)$, $-x \cdot sin(t) + y \cdot cos(t) = f'(t)$, and $x \cdot cos(t) + y \cdot sin(t) + f''(t) \neq 0$. Prove: $(\partial_x z)^2 + (\partial_y z)^2 = z^2$.
(b) Verify that the system $x = s^p + t^p$, $y = s^{p+1} + t^{p+1}$, $z = s^{p+2} + t^{p+2}$ defines the real-analytic function $z(x, y)$ (for which values of s, t?) Compute $\partial_x z$, $\partial_y z$.
(c) Consider the map $\mathbb{C} \ni z \to exp(z) \in \mathbb{C}$. Identify $\mathbb{C} \cong \mathbb{R}^2$, by $z \to (Re(z), Im(z))$, and denote by $\mathbb{R}^2 \stackrel{[exp]_\mathbb{R}}{\to} \mathbb{R}^2$ the corresponding real function. Prove: this function is C^w-invertible locally at each point of $\mathbb{R}^2$. Is this function globally invertible?
(d) Let $\mathbb{R}^1 \xrightarrow{f} \mathbb{R}^1$ be C^k, with $|f'| < 1$, $1 \leq k \leq \infty$. Prove that the map $\mathbb{R}^2 \ni (x, y) \to (x + f(y), y + f(x)) \in \mathbb{R}^2$ is globally(!) C^k-invertible.
(e) Give an example of a function $\mathbb{R} \xrightarrow{f} \mathbb{R}$ such that $f'|_0 = 0$, but f is C^0-globally invertible.
(f) Prove: if $\mathbb{R} \xrightarrow{f} \mathbb{R}$ is C^1 and $f'|_0 = 0$ then f is not C^1-locally invertible at $x = 0$.

Hint to q.1.f: $f(x,y) = (x-y)^2$.)

Hint to q.1.g: $f(x) = \left\{ \begin{array}{l} \frac{x}{2} + x^2 \sin\frac{1}{x},\ x \neq 0 \\ 0,\ x = 0 \end{array} \right.$.