Geometric Calculus 1, 201.1.1031

Homework 9

Fall 2019 (D.Kerner)

(1) (a) Establish the normal form of a C^k-function, $k \geq 1$, $\mathbb{R}^n \supseteq \mathscr{D}_f \xrightarrow{f} \mathbb{R}^m$:

(i) If $m \leq n$ and $rank[f'|_p] = m$, then in some local (C^k) coordinates at $p \in \mathbb{R}^n$ the function is: $f(\underline{x}) = f(p) + (x_1, \dots, x_m)$. (We did the case $m = 1$ in the class.)

(ii) If $m > n$ and $rank[f'|_p] = n$, then in some local (C^k) coordinates at $p \in \mathbb{R}^n$ and at $f(p) \in \mathbb{R}^m$ the function is: $f(\underline{x}) = (x_1, \dots, x_n, 0, \dots, 0)$.

(b) Open mapping theorem: if $\mathbb{R}^n \supseteq \mathscr{D}_f \xrightarrow{f} \mathbb{R}^m$ is C^1, $m \leq n$, and $rank[f'] = m$ on $\mathscr{D}_f$ then f sends open sets to open sets. Give different proofs (via the implicit function theorem, via the inverse function theorem, via the normal form)

(c) Let $\mathbb{R}^n \supseteq \mathscr{D}_f \xrightarrow{f} \mathbb{R}^m$ be C^1, $m \neq n$, and $rank(f'|_p) = min(m, n)$. Can f be locally injective/surjective at p? (wiki: Peano curve)

(d) Let $\mathbb{R}^n \supseteq \mathscr{D}_f \xrightarrow{f} \mathbb{R}^n$ be C^1, $\mathscr{D}_f$ bounded, and f' non-degenerate on $\mathscr{D}_f$. Disprove: $f(\partial\mathcal{U}) = \partial f(\mathcal{U})$. Does this hold at least locally? (What happens if $\mathscr{D}_f$ is unbounded/f' non-invertible?)

(e) Let $\mathbb{R}^n \supset \mathscr{D}_f \xrightarrow{f} \mathbb{R}$ be C^2 and assume $det[f''|_p] \neq 0$. Prove:

(i) There exists a neighborhood $p \in \mathcal{U} \subset \mathscr{D}_f$ such that the subset $f^{-1}(f(p)) \cap \mathcal{U}$ is path-connected. (Check both the case $f'|_p = 0$ and $f'|_p \neq 0$.)

(ii) Suppose p is a local minimum. There exist local coordinates at $p \in \mathscr{D}_f$ such that for any $0 < \epsilon \ll 1$ $f^{-1}(\epsilon)$ is a sphere. (What happens for local maximum/saddle?)

(2) (a) Find the global min/max of f on $\mathscr{D}_f$ in the following cases. (Why does it exist?)

i. $f(x,y) = \Big(x^2 + 2(x-y)^2 + 3(x+y)^2\Big)^4 \Big(x^2 + y^2\Big)^3, \qquad \mathscr{D}_f = \{x^2 + y^2 = 1\}.$

ii. $f(x,y,z) = x^2 - y^2 + z^2 - z^3$ and $\mathscr{D}_f \subset \mathbb{R}^3$ is defined by $\sqrt{x^2+y^2} \leq z \leq 1 + \sqrt{1 - x^2 - y^2}$.

iii. $f(x,y) = \frac{x^2+6xy+3y^2}{x^2-xy+y^2}$ on $\mathbb{R}^2 \setminus \{(0,0)\}$.

(b) Prove the inequalities using extrema under constraints. When the equalities are realised?

i. Hölder inequality: $|\sum x_i y_i| \leq ||\underline{x}||_p \cdot ||\underline{y}||_q$, for $\frac{1}{p} + \frac{1}{q} = 1$.

ii. Comparison of norms: $||\underline{x}||_q \leq ||\underline{x}||_p \leq n^{\frac{1}{p}-\frac{1}{q}} ||\underline{x}||_q$, for $1 \leq p \leq q$.

iii. Arithmetic/geometric/harmonic means: $\frac{n}{\sum \frac{1}{x_i}} \leq \sqrt[n]{x_1 \cdots x_n} \leq \frac{\sum x_i}{n}$, for $\{x_i > 0\}_i$.

(c) Derive Lagrange's theorem (extrema under constraints) as the corollary of the open mapping theorem.

(3) In each case below explain why the point(s) you have found indeed realize the absolute min/max.

(a) Find the min/max distances from the point $0 \in \mathbb{R}^n$ to the set $\{\underline{x} | \sum \frac{|x_i|^d}{a_i^2} = 1\} \subset \mathbb{R}^n$, $d > 0$.

(b) Find the shortest distance from the set $\{\underline{x} | \prod_{i=1}^n |x_i|^{a_i} = 1\} \subset \mathbb{R}^n$ to $0 \in \mathbb{R}^n$.

(c) Among all the boxes inscribed into $\{x^2 + 2y^2 + 3z^2 = 1\} \subset \mathbb{R}^3$, whose faces are parallel to the coordinate planes, find the one of largest volume.

(4) Given a symmetric matrix, $A = A^t \in Mat_{n \times n}(\mathbb{R})$, define the function $f_A(\underline{x}) = \underline{x} \cdot A \cdot \underline{x}^t$.

(a) Prove: if $\underline{x}_0$ is an extremal point of f_A on $S^{n-1} = \{\underline{x} | |\underline{x}| = 1\} \subset \mathbb{R}^n$ then $A \cdot \underline{x}_0^t \sim \underline{x}_0^t$. Thus A has at least one real eigenvector, denote it by $\vec{v}_1$.

(b) Obtain another eigenvector, $\vec{v}_2 \perp \vec{v}_1$, as the extremal point of f_A on the set $\{\underline{x} | |\underline{x}|{=}1, \ \underline{x} \cdot \vec{v}_1{=}0\}$.

(c) In this way construct an orthonormal basis of $\mathbb{R}^n$ composed of eigenvectors of A. Conclude: A is orthogonally-diagonalizable.