Solutions to the midterm, Hedva.3.EE, 20.12.2019

1. (a) The condition $\vec{v}\cdot(\vec{u}\times\vec{w})=0$ implies: the vectors $\vec{v},\vec{u},\vec{w}$ are linearly dependent. Thus the matrix of the system of linear equations is degenerate (of rank at most two). Thus the system has a non-trivial solution.

(b) As the function is differentiable we have $\partial_{\vec{v}}f|_{x_0}=grad(f)|_{x_0}\cdot\frac{\vec{v}}{||\vec{v}||}$. As $\vec{v}$ is orthogonal to the direction of the maximal growth at x_0 we get: $grad(f)|_{x_0}\cdot\vec{v}=0$. Therefore $\partial_{\vec{v}}f|_{x_0}=0$.

2. As f is a continuous function defined on a compact set we get: $\Gamma_f\subset\mathbb{R}^3$ is a compact subset. And then g (being continuous on a compact set) attains its minimum/maximum, by Weierstrass theorem.

3. Note that the points $(1,0)$, $(-1,0)$ belong to the image $f(S^2)$, e.g. $(1,0)=f(1,0,0)$ and $(-1,0)=f(-1,0,0)$.

As f is continuous, defined on a path-connected set, its image is path-connected. Therefore there exists a (continuous) path from $(1,0)$ to $(-1,0)$, inside the image of f.

4. Observe: as $(x,y)\to(0,0)$ one has $sin(x-y)\cdot sin(x+y)=O(||(x,y)||^2)$. In particular, $\lim\limits_{(x,y)\to(0,0)}f(x,y)=0$. Thus we extend f to the origin by $f(0,0)=0$ and obtain a continuous function. To check its partial derivative we restrict this function to the coordinate axes:

$$f|_{y=0}=\begin{cases}\frac{sin^2(x)}{\sqrt{|x|}}, & x\neq0\\ 0, & x=0\end{cases}, \qquad f|_{x=0}=\begin{cases}-\frac{sin^2(y)}{\sqrt{|y|}}, & y\neq0\\ 0, & x=0\end{cases}.$$

Therefore $\partial_x f|_{(0,0)}=0$ and $\partial_y f|_{(0,0)}=0$, i.e. the extended function possess the first order derivatives at $(0,0)$.

5. As f is continuous on a compact set, its minimum and maximum are achieved, e.g. $min=f(p)$ and $max=f(q)$. Take any (continuous) path, $p\overset{\gamma}{\rightsquigarrow}q$, inside $\overline{Ball_1(0,0)}$. By the intermediate value theorem, the value $min<c<max$ is realized on this path, at some point pt_γ.

By taking infinite number of pairwise non-intersecting paths we get the infinite number of points, $\{pt_\gamma\}_\gamma$ all satisfying: $f(pt_\gamma)=c$.