Solutions to Moed.A, Hedva.3.EE, 06.02.2020

1. (This question is the same as q.4.h from homework 1.)

Given a plane $\{\underline{a} \cdot \underline{x} = 1\} \subset \mathbb{R}^n$ and a point $\underline{x}_0 \in \mathbb{R}^n$, consider the line $\{\underline{x}_0 + t \cdot \underline{a}|\ t \in \mathbb{R}\}$. This line is orthogonal to the plane. It intersects the plane at the moment of time t_0 satisfying $(\underline{x}_0 + t_0 \cdot \underline{a})\underline{a} = 1$. Thus $t_0 = \frac{1-\underline{x}_0 \cdot \underline{a}}{||\underline{a}||^2}$. Thus the point symmetric to $\underline{x}_0$ with respect to this plane is $\underline{x}_0 + 2t_0 \cdot \underline{a} = \underline{x}_0 + 2\underline{a}\frac{1-\underline{x}_0 \cdot \underline{a}}{||\underline{a}||^2}$.

2. Counter-examples: $\mathcal{U} = \{(x, y, z)|\ 1 < x^2 + y^2 < 2\} \subset \mathbb{R}^3$ or $\mathbb{R}^3 \setminus \{\hat{z} - axis\}$.

In both cases $\mathcal{U}$ is open, unbounded, path-connected, not simply connected.

3. (a) $\partial_{\vec{v}} f := \lim\limits_{t\to 0} \frac{f(\underline{x}+t\vec{v})-f(\underline{x})}{t||\vec{v}||}$. The answer $\partial_{\vec{v}} f := \lim\limits_{t\to 0} \frac{f(\underline{x}+t\vec{v})-f(\underline{x})}{t}$ was also accepted.

(b) For each fixed ϕ we have: $\lim_{r\to 0} \frac{r\cdot cos^2(\phi)}{r^4+cos^4(\phi)} \cdot e^{cos(2\phi)} = 0$. Therefore, if we want to have directional derivatives, we must extend the function to the origin by 0. Thus we extend f to $(0, 0)$ by 0.

Fix any direction $\mathbb{R}^2 \ni \vec{v} \neq 0$ and the corresponding angle $\phi_{\vec{v}} \in [0, 2\pi)$. We should check the existence of the limit

$$\lim_{t\to 0} \frac{||t \cdot \vec{v}|| \cdot cos^2(\phi_{\vec{v}}) \cdot e^{cos(2\phi_{\vec{v}})}}{(||t \cdot \vec{v}||^4 + cos^4(\phi_{\vec{v}})) \cdot t \cdot ||\vec{v}||} = \lim_{t\to 0} \frac{|t|}{t} \cdot \frac{cos^2(\phi_{\vec{v}}) \cdot e^{cos(2\phi_{\vec{v}})}}{||t \cdot \vec{v}||^4 + cos^4(\phi_{\vec{v}})}.$$

If $cos(\phi_{\vec{v}}) = 0$ then the limit is zero. Otherwise the limit does not exist, as the limits for $t \to 0^\pm$ are distinct.

4. (a) (This is question 5.d from homework.3)

The functions f_1, f_2 are continuous therefore the subsets $S_1, S_2 \subset \mathbb{R}^3$ are closed. As S_1, S_2 are also bounded these sets are compact. Therefore the (continuous) function $d(s_1, s_2)$ on $S_1 \times S_2$ attains its minimum. Hence the distance $d(S_1, S_2)$ is realized for some points of S_1, S_2.

(b) Suppose s_1, s_2 are the points realizing the minimum of the distance. Then the function $h(\underline{x}) = ||\underline{x} - s_2||^2$ has the (absolute) minimum at the point $\underline{x} = s_1$, under the restriction $f_1(\underline{x}) = 0$. Thus, by Lagrange's theorem, at this point holds:

$$2(s_1 - s_2) = grad_{\underline{x}}\big(||\underline{x} - s_2||^2\big)|_{s_1} \sim grad(f_1)|_{s_1}.$$

As we have seen, $grad(f_1)|_{s_1}$ is the normal vector to the surface S_1 at the point s_1. Thus the vector $s_1 - s_2$ is orthogonal to S_1 at the point s_1. Similarly it is orthogonal to S_2 at the point s_2.

5. (This is based on question 1.c from homework 7.)

To identify/classify the critical point we need only the first/second derivatives. Therefore (as in homeworks) we take Taylor's expansion of $f(x, y, z)$ up to order 2. We get: $f(x, y, z) = \frac{xy+yz+xz}{2} + O(x^3, x^2y, xy^2, y^3)$. Therefore $grad(f)|_{(0,0,0)} = 0$, i.e. the origin is indeed a critical point, and it is enough to classify the critical point of $\frac{xy+yz+xz}{2}$. This can be done in many ways e.g.:

- Restrict to the lines $\{y = x, z = 0\}$ and $\{y = -x, z = 0\}$:
  - $f(x, x, 0) = \frac{x^2}{3-sin(x)-cos(x)} \geq 0$. This vanishes at $x = 0$ and is positive otherwise. Thus $x = 0$ is a minimum.
  - $f(x, -x, 0) = \frac{-x^2}{3-sin(x)-cos(x)} \leq 0$. This vanishes at $x = 0$ and is negative otherwise. Thus $x = 0$ is a maximum.

  Thus $(0, 0, 0)$ is a saddle point.
- The Hessian matrix is $\partial^2 f|_{(0,0,0)} = \frac{1}{2}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. To check whether this is positive/negative definite we can use Sylverster's criterion. The first 2×2 minor is $det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1$. Thus $\partial^2 f|_{(0,0,0)}$ is not positive definite. Similarly, $(-\partial^2 f|_{(0,0,0)})$ is not positive definite.

  Therefore $\partial^2 f|_{(0,0,0)}$ is neither positive nor negative definite. Thus $(0, 0, 0)$ is a saddle point.

6. (This is based on q.3.v from homework 11.)

The condition $\frac{y}{2} \leq z \leq y$ implies $y \geq 0$, thus $|y| = y$. We pass to the iterated integral, and then to polar coordinates:

$$\int_{-1}^{1} \Big(\iint\limits_{\substack{0\leq y^2+z^2\leq x^{\frac{2}{3}} \\ \frac{y}{2}\leq z\leq y}} ye^{x^2} dydz\Big)dx = \int_{-1}^{1} \Big(\iint\limits_{\substack{0\leq r\leq |x|^{\frac{1}{3}} \\ arctan(\frac{1}{2})\leq\phi\leq\frac{\pi}{4}}} r^2 cos(\phi)drd\phi\Big)e^{x^2} dx = (\frac{1}{\sqrt{2}} - sin(arctan(\frac{1}{2}))\int_{-1}^{1} \frac{|x|}{3} e^{x^2} dx =$$

$$= 2\frac{(\frac{1}{\sqrt{2}} - sin(arctan(\frac{1}{2}))}{3} \int_0^1 xe^{x^2} dx = \frac{(\frac{1}{\sqrt{2}} - sin(arctan(\frac{1}{2}))}{3} \cdot (e^1 - 1).$$

One can also compute $sin(arctan(\frac{1}{2}))$, e.g., $sin^2 = \frac{sin^2}{sin^2+cos^2} = \frac{tan^2}{tan^2+1}$, thus $sin(arctan(\frac{1}{2})) = \frac{1}{\sqrt{5}}$.

7. (This is based on question 5 from homework 14.)

(a) By the direct check: $\partial_x F_y = \partial_y F_x$. Therefore the field is locally conservative in $\mathbb{R}^2 \setminus (0, 0)$.

(b) To check that the field is globally conservative we should check: $\oint\limits_{\vec{C}} \vec{F} d\vec{C} = 0$ for any loop in $\mathbb{R}^2 \setminus (0,0)$. As was shown in homework 14, it is enough to check the vanishing of just one integral, for $C = \{x^2 + y^2 = R^2\} \subset \mathbb{R}^2$. This follows also from Green's theorem. We check the vanishing:

$$\oint\limits_{x^2+y^2=R^2} \frac{(y^2 - x^2)dx - 2xydy}{(x^2+y^2)^2} \overset{\substack{x=R\cdot cos(\phi)\\ y=R\cdot sin(\phi)}}{=} \int\limits_0^{2\pi} \frac{-sin^3(\phi) - cos^2(\phi)sin(\phi)}{R} d\phi = 0.$$

Therefore the field is globally conservative in $\mathbb{R}^2 \setminus (0,0)$.

Another solution: Once we check that $\partial_x F_y = \partial_y F_x$, we can try to find the potential. By integrating $\int F_x dx$, $\int F_y dy$ one readily sees that $\vec{F} = grad(\phi)$ for $\phi(x,y) = \frac{x}{x^2+y^2}$. Note that ϕ is well defined (and infinitely differentiable) on the whole $\mathbb{R}^2 \setminus \{0,0\}$. Thus $\vec{F}$ is conservative.

8. The surface is closed and the field is continuously differentiable in the whole $\mathbb{R}^3$. We use the Gauß theorem (note the minus sign due to the inner normal):

$$\iint\limits_{\vec{S}} \vec{F} d\vec{S} = - \iiint\limits_{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\leq 1} div\vec{F} \cdot dxdydz \overset{\substack{x=a\tilde{x}\\ y=b\tilde{y}\\ z=c\tilde{z}}}{=} -abc \iiint\limits_{\tilde{x}^2+\tilde{y}^2+\tilde{z}^2\leq 1} (4x^3 + 8y^7 + 1) d\tilde{x}d\tilde{y}d\tilde{z} = -abc \iiint\limits_{\tilde{x}^2+\tilde{y}^2+\tilde{z}^2\leq 1} 1 d\tilde{x}d\tilde{y}d\tilde{z} = -\frac{4\pi}{3} abc.$$