Solutions to Moed.B, Hedva.3.EE, 27.02.2020

1. The domain can be presented in the form $\{(x-1)^2+(y+1)^2 \le 2\}$.

Solution 1. The function $f(x,y)=|xy|$ satisfies: $f(x,y)=f(-x,y)=f(x,-y)=f(-x,-y)$. Therefore, while searching for the critical points in the interior points, we can assume $x \ge 0$ and $y \ge 0$. The only critical point of xy is $(0,0)$. Note that it lies on the boundary of the ball. To this we must add also the points where f is non-differentiable, these are all the points satisfying $xy=0$. At all such points f vanishes.

When looking for the critical points of f on the boundary of the ball, we again use $f(x,y)=f(-x,y)=f(x,-y)=f(-x,-y)$. Therefore it is enough to check the critical points of xy on the circle $(x-1)^2+(y+1)^2=2$, and also to check the points where the function might have the differentiability problem, i.e. $|xy|=0$. The critical points of xy on the circle $(x-1)^2+(y+1)^2=2$ are obtained in the standard way. (e.g. by the condition $grad(xy) \sim grad((x-1)^2+(y+1)^2-2)$.) One gets the condition $(y-x+1)(y+x)=0$. Together with $(x-1)^2+(y+1)^2=2$ one gets:

- Either $x=-y=0$, with $f(0,0)=0$;
- Or $x=-y=2$, with $f(-2,2)=4$;
- Or $y+1=x$, with $x=\frac{1\pm\sqrt{3}}{2}$. Here: $f(\frac{1\pm\sqrt{3}}{2},\frac{-1\pm\sqrt{3}}{2})=|\frac{-1-3}{2}|=2$.

Thus the minimal value of f is 0, while the maximal is 4.

Solution 2. As $f(x,y)=|xy|$, one has: $f \ge 0$. Thus the minimal value of f is 0 and it is obtained at the set of points where $xy=0$. To find the maximal value of f one can pass to the polar coordinates, then the expression for f is: $r^2|\frac{sin(2\phi)}{2}|$. Note that $|sin(2\phi)| \le 1$ and attains 1 for $\phi=-\frac{\pi}{4}$. In addition, by drawing the circle $\{(x-1)^2+(y+1)^2=2\}$, one gets: the maximal value of r is $2\sqrt{2}$ and is attained for $\phi=-\frac{\pi}{4}$. Altogether we get: $f(x,y)\le 4$ and this value is achieved at the point $(2,-2)$.

2. A counterexample: $f(x,y,z)=x+y+z$.

More generally, if f satisfies the assumptions then by the implicit function theorem we have: $\frac{\partial y}{\partial x}=-\frac{\partial_x f}{\partial_y f}$, $\frac{\partial z}{\partial y}=-\frac{\partial_y f}{\partial_z f}$, $\frac{\partial x}{\partial z}=-\frac{\partial_z f}{\partial_x f}$. Thus $\frac{\partial y}{\partial x}\cdot\frac{\partial z}{\partial y}\cdot\frac{\partial x}{\partial z}=-1$. Therefore any f that satisfies the assumptions gives a counterexample.

3. We pass from the triple integral to the repeated integral, projecting onto the yz-plane. The projection is the domain bounded by two ellipses. Thus we have:

$$\iiint\limits_V \frac{x}{1+x^2}dxdydz = \iint\limits_{\{\frac{1}{2}\le 4y^2+9z^2\le 1\}} \Big(\int\limits_0^{\sqrt{1-4y^2-9z^2}} \frac{x}{1+x^2}dx\Big)dydz = \iint\limits_{\{\frac{1}{2}\le 4y^2+9z^2\le 1\}} \frac{ln(2-4y^2-9z^2)}{2}dydz \overset{\substack{\tilde{y}=2y\\ \tilde{z}=3z}}{=\!=} \tag{1}$$

$$= \iint\limits_{\{\frac{1}{2}\le \tilde{y}^2+\tilde{z}^2\le 1\}} \frac{ln(2-\tilde{y}^2-\tilde{z}^2)}{2\cdot 2\cdot 3}d\tilde{y}d\tilde{z} = 2\pi \int\limits_{\{\frac{1}{\sqrt{2}}\le r\le 1\}} \frac{ln(2-r^2)}{2\cdot 2\cdot 3}r\cdot dr \overset{t=r^2}{=\!=} \frac{2\pi}{24}\int\limits_{\frac{1}{2}}^{1} ln(2-t)dt = \frac{2\pi}{24}\left(\frac{3}{2}ln(\frac{3}{2})+\frac{1}{2}\right).$$

4. By the assumptions the surface S is smooth at each point, and its normal is $\nabla(f)$. Therefore:

$$\iint\limits_S \nabla(f)\cdot d\vec{S} = \iint\limits_S \nabla(f)\cdot\frac{\nabla(f)}{||\nabla(f)||}dS = \iint\limits_S ||\nabla(f)||\cdot dS.$$

Here $||\nabla(f)||$ is a continuous positive function on S. The area of S is positive, as S is smooth. Therefore $\iint\limits_S ||\nabla(f)||\cdot dS > 0$.

5. By the direct check, the field $\vec{F}=\frac{(y,-x)}{x^2+y^2}$ is locally conservative. Therefore $\int\limits_C \vec{F}\cdot d\vec{C}+\int\limits_{C_1}\vec{F}\cdot d\vec{C}_1+\int\limits_{C_2}\vec{F}\cdot d\vec{C}_2+\int\limits_{C_3}\vec{F}\cdot d\vec{C}_3=0$, where: $C_1=\{x=0,\ y\in[-\sqrt[10]{100},-r]\}$, $C_2=\{x=0,\ y\in[r,\sqrt[10]{100}]\}$, $C_3=\{x^2+y^2=r^2,\ x\le 0\}$.

(Note that the curves C,C_1,C_2,C_3 bound a simply connected region.)

Note that $\vec{F}|_{x=0}=\frac{(y,0)}{y^2}$. Therefore $\int\limits_{C_1}\vec{F}\cdot d\vec{C}_1=0=\int\limits_{C_2}\vec{F}\cdot d\vec{C}_2$.

Then $\int\limits_C \vec{F}\cdot d\vec{C}=-\int\limits_{C_3}\vec{F}\cdot d\vec{C}_3$, here C_3 is oriented clockwise. We get:

$$\int\limits_C \vec{F}\cdot d\vec{C} = -\int\limits_{C_3}\frac{(y,-x)}{x^2+y^2}\cdot d\vec{C}_3 = \int\limits_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{(r\cdot sin(\phi),-r\cdot cos(\phi))}{r^2}\cdot(-r\cdot sin(\phi),r\cdot cos(\phi))d\phi = -\pi.$$

6. *Solution 1.* Think of the curve C as the (oriented) boundary of the surface $S=\{x^2+y^2+z^2\le 1,\ y+z=-1\}$. This surface is a disc. The (prescribed) orientation of the curve determines the direction of the normal to the disc: $\mathcal{N}=(0,1,1)$. The field has continuous partial derivatives on S, therefore we can use Stokes theorem. Note that $rot(\vec{F}) = (1,1,1)$ thus we have:

$$\oint\limits_C \vec{F}\cdot d\vec{r} = \iint\limits_S (1,1,1)\cdot d\vec{S} = \iint\limits_S (1,1,1)\cdot \frac{(0,1,1)}{|(0,1,1)|} dS = \sqrt{2}\iint\limits_S 1\cdot dS = \sqrt{2}\cdot\big(\text{area of } S\big).$$

The area of S can be computed in various ways. For example, its diameter is the distance between the points $(0,0,-1)$, $(0,-1,0)$, hence the radius equals $\frac{1}{\sqrt{2}}$. Altogether we get:

$$\oint\limits_C \vec{F}\cdot d\vec{r} = \sqrt{2}\cdot\pi\cdot\frac{1}{2}.$$

<u>*Solution 2.*</u> We use the following parametrization of the curve: $\gamma(t) = (x(t),y(t),z(t)) = \left(\frac{cos(t)}{\sqrt{2}}, \frac{sin(t)-1}{2}, -\frac{sin(t)+1}{2}\right)$.

Note that $\gamma'(t) = \left(-\frac{\sin(t)}{\sqrt{2}}, \frac{\cos(t)}{2}, -\frac{\cos(t)}{2}\right)$ and $F(\gamma(t)) = \left(-\sin(t)-1, \frac{\cos(t)}{\sqrt{2}}, \frac{\cos(t)}{\sqrt{2}} + \frac{\sin(t)-1}{2}\right)$.

Therefore the scalar product of these two vectors is: $F(\gamma(t))\cdot\gamma'(t) = \frac{\sin^2(t)}{\sqrt{2}} + \frac{\sin(t)}{\sqrt{2}} - \frac{\sin(t)\cos(t)}{4} + \frac{\cos(t)}{4}$.

Substitute this into the integral: $\oint\limits_C \vec{F}\cdot d\vec{r} = \int\limits_{t=0}^{2\pi} F(\gamma(t))\cdot\gamma'(t)\,dt = \ldots$.

The integral over $\sin(t)$ vanishes, and so do the integrals over $\cos(t)$ and $\sin(t)\cos(t) = \frac{\sin(2t)}{2}$. We're finally left with

$$\oint\limits_C \vec{F}\cdot d\vec{r} = \ldots = \int_{t=0}^{2\pi}\frac{\sin^2(t)}{\sqrt{2}}\,dt = \frac{1}{2\sqrt{2}}\int_{t=0}^{2\pi} 1-\cos(2t)\,dt = \frac{\pi}{\sqrt{2}}.$$