Solutions to Moed.C, Hedva.3.EE, 16.07.2020

1. As $\vec{v}\cdot(\vec{u}\times\vec{w})\neq 0$ the vectors $\vec{v},\vec{u},\vec{w}$ are linearly independent (do not lie in one plane). Therefore their end-points are the vertices of a triangle, while $||\vec{v}-\vec{u}||$, $||\vec{v}-\vec{w}||$, $||\vec{u}-\vec{w}||$ are the lengths of the sides of this triangle.

This triangle is non-degenerate, i.e. the vertices do no lie on one line. Therefore none of $||\vec{v}-\vec{u}||$, $||\vec{v}-\vec{w}||$, $||\vec{u}-\vec{w}||$ is the sum of two others.

2. The function $f(x,y,z)$ is periodic in z, with the period 2π. Therefore instead of considering the unbounded set $\partial\mathcal{U}\subset\mathbb{R}^3$ we can restrict to $\partial\mathcal{U}\cap\{0\le z\le 2\pi\}\subset\mathbb{R}^3$. This set is closed and bounded, thus compact. Therefore f attains its minimum/maximum on $\partial\mathcal{U}\cap\{0\le z\le 2\pi\}$, and hence on $\partial\mathcal{U}$ as well.

As $f(x,y,z)=g(x,y)+sin(z)$, the minimum of f occurs when $z\in-\frac{\pi}{2}+2\pi\mathbb{Z}$ and $g(x,y)$ has the minimum. The maximum occurs when $z\in\frac{\pi}{2}+2\pi\mathbb{Z}$ and $g(x,y)$ has the maximum. Therefore the question is reduced to min/max of $g(x,y)=x^2+y^2$ on the boundary of the subset $\{\frac{x^2}{a^2}+\frac{y^2}{b^2}<1,\ \frac{x^2}{b^2}+\frac{y^2}{a^2}<1\}\subset\mathbb{R}^2$. This is the intersection of two ellipses, the boundary consists of the four arcs.

- Suppose the arc is a part of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Then g on this arc is: $x^2+b^2(1-\frac{x^2}{a^2})=b^2-x^2(\frac{b^2}{a^2}-1)$. As $b>a$ the minimum of this expression is achieved at $x^2=a^2$, while the maximum at $x=0$.
  Note that the points $(0,\pm b)$ do not satisfy the condition $\frac{x^2}{b^2}+\frac{y^2}{a^2}<1$. The points $(\pm a,0)$ do satisfy it, and $g(\pm a,0)=a^2$.
- Suppose the arc is a part of $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$. Then g on this arc is: $x^2+a^2(1-\frac{x^2}{b^2})=a^2+x^2(1-\frac{a^2}{b^2})$. As $b>a$ the minimum of this expression is achieved at $x^2=0$, while the maximum at $x^2=b^2$.
  Note that the points $(\pm b,0)$ do not satisfy the condition $\frac{x^2}{a^2}+\frac{y^2}{b^2}<1$. The points $(0,\pm a)$ do satisfy it, and $g(0,\pm a)=a^2$.
- It remains to check the intersection points of the two ellipses: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1=\frac{x^2}{b^2}+\frac{y^2}{a^2}$. These satisfy $x^2=y^2=\frac{a^2b^2}{a^2+b^2}$, and $g(*,*)=\frac{2a^2b^2}{a^2+b^2}$.

3. The surface is naturally parameterized by x,y. These parameters vary in the region $\{x^2+y^2+2y\le 2020\}\subset\mathbb{R}^2$, which is a shifted disc: $\{x^2+(y+1)^2\le 2021\}\subset\mathbb{R}^2$. Therefore, for any function h holds:

$$\iint h\cdot dS=\iint h\cdot\sqrt{1+(\frac{\partial z}{\partial x})^2+(\frac{\partial z}{\partial y})^2}dxdy=\iint h\sqrt{2}dxdy.$$

Thus

$$Area=\iint_S 1dS=\iint_{\{x^2+y^2+2y\le 2020\}}\sqrt{2}\cdot dxdy=\sqrt{2}\cdot\Big(\text{area of the disc of radius }\sqrt{2021}\Big)=\sqrt{2}\cdot\pi\cdot(\sqrt{2021})^2.$$

4. See question 3 of Moed.B, 2020.02.27.

5. The integration path bounds the disc, denote it D. We apply Stokes theorem: $\oint_C\vec{F}\cdot d\vec{r}=\iint_D(1,-1,1)\cdot d\vec{S}$. By the assumption on the direction of the path, the (unit) normal to the disc is: $\frac{(0,1,-1)}{\sqrt{2}}$. Therefore:

$$\oint_C\vec{F}\cdot d\vec{r}=-\sqrt{2}\cdot(\text{the area of }D).$$

The diameter of D is obtained as the distance between the points $(0,0,0)$ and $(0,1,1)$, which is $\sqrt{2}$.

Therefore $\oint_C\vec{F}\cdot d\vec{r}=-\sqrt{2}\pi(\frac{1}{\sqrt{2}})^2$.

6. Note that $div(\frac{(x,y,z)}{(x^2+y^2+z^2)^{\frac{3}{2}}})=0$. Therefore we can use Gauss theorem to replace the surface by a simpler one, as far as the field remains differentiable in the whole body of integration.

Solution 1. First we note: $\iint\limits_{\substack{|x|^3+|y|^3+|z|^3=1\\ z\geq 0}} \frac{(x,y,z)\cdot d\vec{S}}{(x^2+y^2+z^2)^{\frac{3}{2}}} = \iint\limits_{\substack{|x|^3+|y|^3+|z|^3=1\\ z\leq 0}} \frac{(x,y,z)\cdot d\vec{S}}{(x^2+y^2+z^2)^{\frac{3}{2}}}$, as both the field and the normal are anti-symmetric under $(x,y,z)\leftrightarrow(-x,-y,-z)$. Therefore

$$\iint\limits_{\substack{|x|^3+|y|^3+|z|^3=1\\ z\geq 0}} \frac{(x,y,z)}{(x^2+y^2+z^2)^{\frac{3}{2}}}\cdot d\vec{S} = \frac{1}{2}\oint\!\!\!\oint\limits_{|x|^3+|y|^3+|z|^3=1} \frac{(x,y,z)}{(x^2+y^2+z^2)^{\frac{3}{2}}}\cdot d\vec{S}.$$

To compute this later integral we would like to use Gauss theorem, but the field is not differentiable at $(0,0,0)$. Thus we use Gauss theorem for the body: $\{|x|^3+|y|^3+|z|^3\leq 1,\ x^2+y^2+z^2\geq\epsilon^2\}$. Here $0<\epsilon<1$ is a (small) constant. As $div(\frac{(x,y,z)}{(x^2+y^2+z^2)^{\frac{3}{2}}})=0$ we have:

$$\frac{1}{2}\oint\!\!\!\oint\limits_{|x|^3+|y|^3+|z|^3=1} \frac{(x,y,z)}{(x^2+y^2+z^2)^{\frac{3}{2}}}\cdot d\vec{S} = -\frac{1}{2}\oint\!\!\!\oint\limits_{x^2+y^2+z^2=\epsilon^2} \frac{(x,y,z)}{(x^2+y^2+z^2)^{\frac{3}{2}}}\cdot d\vec{S},\quad \text{with the inner normal.}$$

For the later integral we note that $x^2+y^2+z^2\equiv\epsilon^2$ along the surface, thus

$$-\frac{1}{2}\oint\!\!\!\oint\limits_{x^2+y^2+z^2=\epsilon^2} \frac{(x,y,z)}{(x^2+y^2+z^2)^{\frac{3}{2}}}\cdot d\vec{S} = -\frac{1}{2}\oint\!\!\!\oint\limits_{x^2+y^2+z^2=\epsilon^2} \frac{(x,y,z)}{\epsilon^3}\cdot d\vec{S} \overset{Gauss}{=\!=} \frac{1}{2\epsilon^3}\iiint\limits_{x^2+y^2+z^2\leq\epsilon^2} 3dV = \frac{4\pi}{2}.$$

Solution 2. We would like to replace the initial surface by the planar domain $\{|x|^3+|y|^3\leq 1,\ z=0\}$. But this domain contains the point $(0,0,0)$, where the field is not differentiable. Thus we replace the initial surface by the union:

$$\underbrace{\left\{\begin{array}{l} |x|^3+|y|^3\leq 1,\ z=0\\ x^2+y^2\geq\epsilon^2\end{array}\right\}}_{S_1}\cup\underbrace{\{x^2+y^2+z^2=\epsilon^2, z\geq 0\}}_{S_2}.$$

As $div(\frac{(x,y,z)}{(x^2+y^2+z^2)^{\frac{3}{2}}})=0$ we have by Gauss theorem:

$$\iint\limits_{\substack{|x|^3+|y|^3+|z|^3=1\\ z\geq 0}} \frac{(x,y,z)\cdot d\vec{S}}{(x^2+y^2+z^2)^{\frac{3}{2}}} + \iint\limits_{S_1}\frac{(x,y,z)\cdot d\vec{S}}{(x^2+y^2+z^2)^{\frac{3}{2}}} + \iint\limits_{S_2}\frac{(x,y,z)\cdot d\vec{S}}{(x^2+y^2+z^2)^{\frac{3}{2}}} = 0.$$

In the later two integrals the normals are taken downstairs.

The normal to S_1 is $(0,0,-1)$, therefore for the integral over S_1 we have: $(x,y,z)\cdot d\vec{S}=-zdS=0$, as $z=0$. Thus

$$\iint\limits_{\substack{|x|^3+|y|^3+|z|^3=1\\ z\geq 0}} \frac{(x,y,z)\cdot d\vec{S}}{(x^2+y^2+z^2)^{\frac{3}{2}}} = \iint\limits_{S_2}\frac{(x,y,z)\cdot d\vec{S}}{(x^2+y^2+z^2)^{\frac{3}{2}}},\quad \text{with the normal upstairs.}$$

The later integral is computed in various ways, e.g.

$$\iint\limits_{S_2}\frac{(x,y,z)\cdot d\vec{S}}{(x^2+y^2+z^2)^{\frac{3}{2}}} = \iint\limits_{S_2}\frac{\vec{r}\cdot\frac{\vec{r}}{r}dS}{r^3} = \iint\limits_{S_2}\frac{dS}{r^2} = \frac{1}{\epsilon^2}\iint\limits_{S_2} dS = \frac{\text{area of } S_2}{\epsilon^2} = \frac{1}{\epsilon^2}2\pi\epsilon^2.$$