HOMEWORK SHEET 1: SOLUTIONS

INTRODUCTION TO COMPLEX ANALYSIS FOR ELECTRIC ENGINEERING

1. Question 1

a) For every $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$, we know that

$$cos(n\alpha) + isin(n\alpha) = e^{n\alpha i} = (cos(\alpha) + isin(\alpha))^n = \sum_{k=0}^{n} \binom{n}{k} i^k \cos^{n-k}(\alpha)\sin^k(\alpha).$$

If $n = 4$, we get

$$\begin{aligned} cos(4\alpha) = Re(e^{4\alpha i}) = \binom{4}{0}\cos^4(\alpha) + \binom{4}{2} i^2 \cos^2(\alpha)\sin^2(\alpha) + \binom{4}{4} i^4 \sin^4(\alpha) \\ = \cos^4(\alpha) - 6\cos^2(\alpha)\sin^2(\alpha) + \sin^4(\alpha), \end{aligned}$$

while for $n = 5$ we get

$$\begin{aligned} \sin(5\alpha) = Im(e^{5\alpha i}) = \frac{1}{i}\left[\binom{5}{1} i \cos^4(\alpha)\sin(\alpha) + \binom{5}{3} i^3 \cos^2(\alpha)\sin^3(\alpha) + \binom{5}{5} i^5 \sin^5(\alpha)\right] \\ = 5\cos^4(\alpha)\sin(\alpha) - 10\cos^2(\alpha)\sin^2(\alpha) + \sin^5(\alpha). \end{aligned}$$

b) For every $n \geq 0$ and $\phi \in \mathbb{R}$:

$$\sum_{k=0}^{n} cos(k\phi) = \sum_{k=0}^{n} Re(e^{ik\phi}) = Re\left(\sum_{k=0}^{n} e^{ik\phi}\right) = Re\left(\sum_{k=0}^{n} (e^{i\phi})^k\right)$$

and

$$\sum_{k=0}^{n} sin(k\phi) = \sum_{k=0}^{n} Im(e^{ik\phi}) = Im\left(\sum_{k=0}^{n} e^{ik\phi}\right) = Im\left(\sum_{k=0}^{n} (e^{i\phi})^k\right).$$

The last sum is a trigonometric sum and hence we have the formula

$$\begin{aligned} \sum_{k=0}^{n} (e^{i\phi})^k = \frac{1 - (e^{i\phi})^{n+1}}{1 - e^{i\phi}} = \frac{(1 - (e^{i\phi})^{n+1})(1 - e^{-i\phi})}{(1 - e^{i\phi})(1 - e^{-i\phi})} \\ = \frac{1 - e^{-i\phi} - e^{i(n+1)\phi} + e^{in\phi}}{2 - (e^{i\phi} + e^{-i\phi})} = \frac{1 - e^{-i\phi} - e^{i(n+1)\phi} + e^{in\phi}}{2 - 2cos(\phi)}, \end{aligned}$$

so

$$\sum_{k=0}^{n} cos(k\phi) = Re\left(\frac{1 - e^{-i\phi} - e^{i(n+1)\phi} + e^{in\phi}}{2 - 2cos(\phi)}\right) = \frac{1 - cos(\phi) - cos((n+1)\phi) + cos(n\phi)}{2 - 2cos(\phi)}$$

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and

$$\sum_{k=0}^{n} sin(k\phi) = Im\left(\frac{1-e^{-i\phi}-e^{i(n+1)\phi}+e^{in\phi}}{2-2cos(\phi)}\right) = \frac{sin(\phi)-sin((n+1)\phi)+sin(n\phi)}{2-2cos(\phi)}.$$

c) Let $p(x) \in \mathbb{R}[x]$, so $p(x) = a_0 + a_1x + \ldots + a_nx^n$, where $a_0, a_1, \ldots, a_n \in \mathbb{R}$. If $z_0 \in \mathbb{C}$, then

$$p(z_0) = 0 \iff \sum_{k=0}^{n} a_k z_0^k = 0 \iff \sum_{k=0}^{n} a_k \overline{z_0}^k = \sum_{k=0}^{n} \overline{a_k} \cdot \overline{z_0}^k = \overline{\sum_{k=0}^{n} a_k z_0^k} = 0 \iff p(\overline{z_0}) = 0.$$

2. Question 2:

a) Let $a \in \mathbb{C}$ and define the mapping $\psi_a : \mathbb{C} \to \mathbb{C}$ by $\psi_a(z) = a \cdot z$. Write a in its Cartesian representation

$$a = x_a + iy_a, \quad x_a, y_a \in \mathbb{R}$$

then for every $z = x + iy \in \mathbb{C}$:

$$\psi_a(z) = (x_a + iy_a)(x + iy) = (x_a x - y_a y) + i(x_a y + y_a x).$$

Thus $\psi_a : \mathbb{R}^2 \to \mathbb{R}^2$ is the following

$$\psi_a(x, y) = (x_a x - y_a y, x_a y + y_a x), \quad \forall (x, y) \in \mathbb{R}^2$$

and the representative matrix of ψ_a with respect to the standard basis in $\mathbb{R}^2$ is

$$A := [\psi_a] = \begin{pmatrix} x_a & -y_a \\ y_a & x_a \end{pmatrix}.$$

• Suppose $|a| = 1$, thus

$$AA^t = \begin{pmatrix} x_a & -y_a \\ y_a & x_a \end{pmatrix} \begin{pmatrix} x_a & y_a \\ -y_a & x_a \end{pmatrix} = \begin{pmatrix} x_a^2 + y_a^2 & 0 \\ 0 & y_a^2 + x_a^2 \end{pmatrix} = \begin{pmatrix} |a|^2 & 0 \\ 0 & |a|^2 \end{pmatrix} = I_2$$

and also

$$\det(A) = x_a^2 + y_a^2 = |a|^2 = 1.$$

• As $|a| = 1$, a is of the form $a = e^{i\theta_a}$ for some $\theta_a \in [0, 2\pi)$. Thus for every $z = re^{i\theta} \in \mathbb{C}$:

$$\psi_a(z) = e^{i\theta_a} \cdot re^{i\theta} = re^{i(\theta+\theta_a)}$$

so $\psi_a(z)$ just rotates the point z by θ_a degrees, while keeping the same distance from 0.

b) Define the conjugation mapping $\tau : \mathbb{C} \to \mathbb{C}$, by $\tau(z) = \overline{z}$. It is easily seen that

$$\tau(z_1 + z_2) = \overline{z_1 + z_2} = \overline{z_1} + \overline{z_2} = \tau(z_1) + \tau(z_2), \quad \forall z_1, z_2 \in \mathbb{C}$$

and

$$\tau(\alpha z) = \overline{\alpha z} = \overline{\alpha}\tau(z), \quad \forall \alpha \in \mathbb{C}, z \in \mathbb{C},$$

so τ is is a linear mapping over the field $\mathbb{R}$, but not over $\mathbb{C}$.

• For every $z = x + iy \in \mathbb{C}$, we have $\tau(z) = x - iy$, so $\tau : \mathbb{R}^2 \to \mathbb{R}^2$ is of the form

$$\tau(x, y) = (x, -y)$$

so its representative matrix of τ with respect to the standard basis in $\mathbb{R}^2$ is

$$B := [\tau] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$B \cdot B^t = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = I_2 \text{ and } \det(B) = -1.$$

• For every $z = re^{i\theta} \in \mathbb{C}$, we have $\tau(z) = re^{-i\theta}$ so this is the point which is symmetric to z with respect to the $x-$axis (the real line).

3. Question 3:

Let (z_n) be a sequence in $\mathbb{C}$ and $z \in \mathbb{C}$. Recall that $z_n \to z$ if and only if $|z_n - z| \to 0$.

a)(i) True:

$$\begin{aligned} z_n \to z &\iff \big|Re(z_n) - Re(z) + i(Im(z_n) - Im(z))\big| = |z_n - z| \to 0 \\ &\iff (Re(z_n) - Re(z))^2 + (Im(z_n) - Im(z))^2 \to 0. \end{aligned}$$

Now, it is not hard to see that for every two sequences of real numbers (x_n) and (y_n), we have

$$x_n^2 + y_n^2 \to 0 \iff x_n \to 0 \text{ and } y_n \to 0.$$

One direction is simply arithmetics, while the other one follows from the Sandwich rule and the fact that $0 \leq x_n^2, y_n^2 \leq x_n^2 + y_n^2$. Therefore,

$$\begin{aligned} z_n \to z &\iff Re(z_n) - Re(z) \to 0 \text{ and } Im(z_n) - Im(z) \to 0 \\ &\iff Re(z_n) \to Re(z) \text{ and } Im(z_n) \to Im(z) \\ &\iff (Re(z_n), Im(z_n)) \to (Re(z), Im(z)). \end{aligned}$$

b)(i) • If $|z| < 1$, then $\lim_{n\to\infty} |z^n - 0| = \lim_{n\to\infty} |z|^n = 0$, so $\lim_{n\to\infty} z^n = 0$.
• If $|z| > 1$, then $\lim_{n\to\infty} |z^n| = \lim_{n\to\infty} |z|^n = \infty$ and $\lim_{n\to\infty} z^n = \infty$.
• If $|z| = 1$ and suppose that $z^n \to \ell$, then clearly $z^{n+1} \to \ell$ and by arithmetics $z^{n+1} \to z \cdot \ell$. By the uniqueness of the limit (of a converging sequence) we get $z\ell = \ell$, but $z^n \to \ell$ implies that $1 = |z^n| \to |\ell|$, i.e., that $|\ell| = 1 \Longrightarrow \ell \neq 0$, so $z = 1$. So: if $z = 1$, then $\lim_{n\to\infty} z^n = 1$ and if $|z| = 1, z \neq 1$, then z^n does not converge.

b)(iii) We show the limit does not exist:

$$f(z) = \frac{z^2 - \overline{z}^2}{z^2 + \overline{z}^2} = \frac{z^2 - \overline{z^2}}{z^2 + \overline{z^2}} = \frac{2iIm(z^2)}{2Re(z^2)} = i\frac{Im(z^2)}{Re(z^2)},$$

so if $z_n = n \to \infty$ then $f(z_n) = 0$. On the other hand, if $w_n = n + in \to \infty$ then $f(w_n) = i$. Therefore the limit $\lim_{z\to\infty} f(z)$ does not exist.

b)(iv) Let

$$f(z) = \frac{Re(z) \cdot Im(z)^2}{Re(z)^2 + Im(z)^4}.$$

Let $z_n = \frac{1}{n} \to 0$, then $f(z_n) = 0$. On the other hand, let $w_n = \frac{1}{n^2} + i\frac{1}{n} \to 0$, then

$$f(w_n) = \frac{\frac{1}{n^2} \cdot \frac{1}{n^2}}{\frac{1}{n^4} + \frac{1}{n^4}} = \frac{1}{2},$$

therefore the limit $\lim_{z\to 0} f(z)$ does not exist.

4. Question 4:

Recall that a set $X \subset \mathbb{C}$ is **open** if and only if for every $z \in X$, there exists $\epsilon > 0$ such that $Ball_\epsilon(z) \subset X$. A set $X \subset \mathbb{C}$ is closed if $\partial X \subset X$.

b) True: A set $X \subset \mathbb{C}$ is open if and only if $\mathbb{C} \setminus X$ is closed.

• Assume X is open. Let $z \in \partial(\mathbb{C} \setminus X)$ and suppose that $z \notin \mathbb{C} \setminus X$, i.e., that $z \in X$. As X is open, there exists $\epsilon > 0$ such that $Ball_\epsilon(z) \subset X$, so

$$Ball_\epsilon(z) \cap (\mathbb{C} \setminus X) = \emptyset \Longrightarrow z \notin \partial(\mathbb{C} \setminus X)$$

and this is a contradiction. So $\partial(\mathbb{C} \setminus X) \subset \mathbb{C} \setminus X$, meaning that $\mathbb{C} \setminus X$ is closed.

• Assume $\mathbb{C} \setminus X$ is closed. Let $z \in X$, so $z \notin \mathbb{C} \setminus X$ and since $\partial(\mathbb{C} \setminus X) \subset \mathbb{C} \setminus X$, we know that $z \notin \partial(\mathbb{C} \setminus X)$. Then there exists $\epsilon > 0$ such that

$$Ball_\epsilon(z) \cap (\mathbb{C} \setminus X) = \emptyset \text{ or } Ball_\epsilon(z) \cap X = \emptyset,$$

but $z \in Ball_\epsilon(z) \cap X$, which implies that

$$Ball_\epsilon(z) \cap (\mathbb{C} \setminus X) = \emptyset \Longrightarrow Ball_\epsilon(z) \subset X,$$

i.e., we shoed that X is open.

c)(i) • As $X \subset \mathbb{C}$, it is easily seen that X is open in $\mathbb{C}$ if and only if X is open in $\bar{\mathbb{C}}$.
• However, this is not true for closed sets: Let

$$X = \mathbb{R}_{\geq 0} = \{z \in \mathbb{C} : Im(z) = 0, Re(z) \geq 0\}.$$

The set X is closed in $\mathbb{C}$, as $\mathbb{C} \setminus X$ is open in $\mathbb{C}$, but X is not closed in $\bar{\mathbb{C}}$: for every $r > 0$, we have

$$\{z : |z| > r\} \cap X \neq \emptyset \text{ and } \{z : |z| > r\} \cap (\bar{\mathbb{C}} \setminus X) \neq \emptyset,$$

which means that $\infty \in \partial(X)$, but $\infty \notin X$, hence X is not closed in $\bar{\mathbb{C}}$.
• A much simpler example is $X = \mathbb{C}$: which is closed in $\mathbb{C}$ but not in $\bar{\mathbb{C}}$.