HW10: SOLUTIONS

1. Question 1.

a) Let $f \in \mathcal{O}(\mathbb{C})$ be $1-1$. Let us explore the behavior of f at ∞. If $z = \infty$ is an essential singularity of f, then $g(z) = f(1/z)$ has an essential singularity at $z = 0$ and g is analytic in $\mathbb{C} \setminus \{0\}$, since f is entire. From the Picard's theorem we must have

$$g(Ball_1(0)^*) = \mathbb{C} \text{ or } \mathbb{C} \setminus \{z_0\} \text{ for some } z_0 \in \mathbb{C}.$$

Therefore, g is not $1-1$ (it is enough to take 2 more points outside the ball to see that) and hence f is not $1-1$, which is a contradiction.

So, $z = \infty$ is not an essential singularity of f, hence $\lim_{z\to\infty} f(z)$ exists (finite of ∞) and by question 2-d we must have that f is a polynomial. As f is $1-1$, it must have only one zero, so it is of the form $f(z) = a_0(z - z_0)^m$. However, if $m > 1$, we can take

$$z_1 = z_0 + e^{2\pi i/m},\ z_2 = z_0 + e^{4\pi i/m}$$

satisfying: $z_1 \neq z_2$ but $f(z_1) = f(z_2)$, which contradicts the assumption. So $m \leq 2$, as required.

b) Let $f, g \in \mathcal{O}(\mathbb{C})$ such that $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. If $f \equiv 0$, then setting $c = 0$ yields $f(z) = cg(z)$. Otherwise, let z_0 be a a zero of $g(z)$, say of order m, thus $f(z_0) = 0$ as well; say z_0 is a zero of $f(z)$ of order n, thus we have

$$g(z) = (z - z_0)^m \widetilde{g}(z),\ f(z) = (z - z_0)^n \widetilde{f}(z)$$

where $\widetilde{g}, \widetilde{f} \in \mathcal{O}(\mathbb{C})$ and $\widetilde{f}(z_0), \widetilde{g}(z_0) \neq 0$. If $n < m$, we get

$$|f(z)| \leq |g(z)| \Longrightarrow |(z - z_0)^n \widetilde{f}(z)| \leq |(z - z_0)^m \widetilde{g}(z)| \Longrightarrow |\widetilde{f}(z)| \leq |(z - z_0)^{m-n} \widetilde{g}(z)|$$

but for $z = z_0$ we get $|\widetilde{f}(z_0)| \leq 0 \Longrightarrow \widetilde{f}(z_0) = 0$ which is a contradiction. So we must have $n \geq m$. Define the function

$$h(z) = \frac{f(z)}{g(z)},$$

we showed that for every z_0 that is a zero of $g(z)$ we must have that z_0 is also a zero of $f(z)$ with $ord_f(z_0) \geq ord_g(z_0)$, thus z_0 is a removable singularity point of $h(z)$; thus h is analytic in except for removable points, hence h is entire and also bounded by 1, therefore $h \equiv c$ with $|c| \leq 1$.

d) No! If such a function exists, then a pole at ∞ means that $\lim_{z\to\infty} f(z) = \infty$, which is equivalent to

$$\lim_{z\to 0} f(1/z) = \infty.$$

However, there exists a sequence of points $z_n = 1/n \to 0$ such that $f(1/z_n) = 0$, which is a contradiction to the limit being equal to ∞.

f) Suppose $z = z_0$ is an essential singularity point of f, thus using Picard's theorem on $Ball_{\epsilon/2}(z_0)^*$, we get that f is not $1-1$ in $Ball_\epsilon(z_0)^*$. Therefore, $z = z_0$ must be a pole of f of order $m \geq 2$ or a zero of order $m \geq 2$.

• If $z = z_0$ is a zero of f of order $m \geq 2$, then one can write $f(z) = (z-z_0)^m g(z)$, where g is analytic in $Ball_\epsilon(z_0)$ and $g(z_0) \neq 0$. Then we must have that $g(z) \neq 0$ for all $z \in Ball_\delta(z_0)$ for some $\delta \leq \epsilon$, and moreover that $g(Ball_\delta(z_0)) \subseteq Ball_\alpha(g(z_0))$ for some $\alpha > 0$. One can choose δ small enough such that $0 \notin Ball_\alpha(g(z_0))$, thus we can define the analytic function $\sqrt[m]{g(z)}$ that is analytic in $Ball_\delta(z_0)$. Define

$$h(z) := (z - z_0)\sqrt[m]{g(z)},$$

that is analytic in $Ball_\delta(z_0)$, satisfying $h(z)^m = f(z)$ for any $z \in Ball_\delta(z_0)$. Fix a point $z_1 \in Ball_\delta(z_0)^*$, so from the open mapping theorem we know that there exists a point $z_2 \in Ball_\delta(z_0)$ such that $h(z_2) = e^{2\pi i/m}h(z_1)$; thus we have

$$f(z_2) = h(z_2)^m = e^{2\pi i}h(z_1)^m = f(z_1)$$

and as $m \geq 2$ we have $z_1 \neq z_2$, as $h(z_2) \neq h(z_1)$. So f is not $1-1$.

• Finally, if $z = z_0$ is a pole of f of order $m \geq 2$, thus $\lim_{z\to z_0} f(z) = \infty$ which implies that there exists $\beta \leq \epsilon$ such that $f(z) \neq 0$ for all $z \in Ball_\beta(z_0)^*$. Then define

$$\widetilde{f}(z) := \frac{1}{f(z)}$$

that is analytic in $Ball_\beta(z_0)^*$ and it has a removable singularity at $z = z_0$; moreover, it has a zero of order m at $z = z_0$. Now, we can use what we already proved for $\widetilde{f}$ and that is that $\widetilde{f}$ is not $1-1$, which clearly implies that f is not $1-1$.

2. Question 2.

a) If the image of f is not dense in $\mathbb{C}$, it means there exists $\omega \in \mathbb{C}$ and $\epsilon > 0$ such that $|f(z) - \omega| > \epsilon$, for every z in the domain of f. Define

$$g(z) := \frac{1}{f(z) - \omega}.$$

Clearly g is analytic at all points z where f is analytic; moreover, if z_0 is a singularity point of f, it must be a pole (as f is meromorphic) so $\lim_{z\to z_0} f(z) = \infty$ and hence $\lim_{z\to z_0} g(z) = 0$, meaning that z_0 is a removable singularity point of g. So g is entire (analytic in $\mathbb{C}$ except for removable singularities) and also bounded, as $|g(z)| < \frac{1}{\epsilon}$, therefore by Liouville's theorem g is constant and then it is easily seen that f is constant as well; contradiction to the assumption that f is not.

b) Suppose f is meromorphic in $\mathcal{U}$, $\mathcal{U}$ is bounded and suppose that $x \subset \mathcal{U}$ is closed. So X is closed and bounded!

If f has infinitely many zeros in X, denoted $(z_n)_{n=0}^\infty$, then this sequence has a subsequence that converges (using the B–W theorem as X is closed and bounded) to a point in X, so we can apply the uniqueness theorem to get f should be 0 on all points!

So f must have finitely many zeros in X. Suppose next that f has infinitely many poles in X. Define the function $g(z) := 1/f(z)$, which is meromorphic because the singularities of g are exactly the singularities of f and the zeros of f: the zeros of f become poles of g, while the poles of g become removable singularities of g. So g has only poles, so it is meromorphic. But g has infinitely many zeros (if z_0 is a pole of $f(z)$, then $\lim_{z\to z_0} f(z) = \infty$ which implies that $\lim_{z\to z_0} g(z) = 0$ so

z_0 is removable singularity of g and it is a zero of g) and this is possible (as proved in part a) only when g is constant and then f is also a constant.

c) Let f me meromorphic in $\overline{\mathbb{C}}$. That means that $z = \infty$ is either removable singularity or an isolated pole. If there are infinitely many singularity points of f, say $(z_n)_{n=0}^{\infty}$, then we have two cases: If (z_n) is bounded, then it has a subsequence which converge to say $\omega \in \mathbb{C}$, which makes ω a non-isolated singularity point of f, that is a contradiction as f is meromorphic.

Otherwise, (z_n) is not bounded, which means it has a subsequence that converges to ∞, which makes $z = \infty$ a non-isolated singularity point of f; contradiction to f being meromorphic once again.

d) See Targil 1 in Tirgul 12.

3. Question 3.

a)i) $f(z) = \frac{z^2+z-1}{z^2(z-1)}$. Singularities are $z = 0, 1$.

•$z_0 = 0$ is a pole of order 2, as $f(z) = \frac{1}{z^2} g_0(z)$ where $g_0(z) = \frac{z^2+z-1}{z-1}$ is analytic at 0; thus

$$Res_{z=0}(f) = \lim_{z \to 0} g_0(z)' = \lim_{z \to 0} \frac{(z-1)(2z+1) - (z^2+z-1)}{(z-1)^2} = 0.$$

• $z_1 = 1$ is a simple pole, as $f(z) = \frac{1}{z-1} g_1(z)$ where $g_1(z) = \frac{z^2+z-1}{z^2}$ is analytic at 1; thus

$$Res_{z=1}(f) = \lim_{z \to 1} g_1(z) = 1.$$

a)ii) $f(z) = z^n \sin(1/z)$. Singularities of f are $z = 0$.

• Around $z_0 = 0$, the Laurent expansion of f at $z = 0$ is

$$f(z) = z^n \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)! z^{2k+1}} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)! z^{2k+1-n}},$$

thus

$$Res_{z=0}(f) = \begin{cases} \frac{(-1)^{n/2}}{(n+1)!} & : 2 \mid n \\ 0 & : 2 \nmid n. \end{cases}$$

• $Res_{z=\infty}(f) = -Res_{z=0}(\frac{1}{z^2} f(1/z)) = -Res_{z=0}(\frac{\sin(z)}{z^{n+2}})$, as

$$\frac{\sin(z)}{z^{n+2}} = \sum_{k=1}^{\infty} \frac{(-1)^k z^{2k+1-n-2}}{(2k+1)!},$$

we get that $2k - n - 1 = -1 \iff 2k = n$, so

$$Res_{z=\infty}(f) = -\begin{cases} \frac{(-1)^{n/2}}{(n+1)!} & : 2 \mid n \\ 0 & : 2 \nmid n. \end{cases}$$

b)i) If $f \in \mathcal{O}(Ball_\epsilon(0))$, then the only singularity of $g(z) := \sin(1/z)f(z)$ in $Ball_{\epsilon/2}(0)$ is $z = 0$; therefore by the residue theorem

$$\frac{1}{2\pi i} \int_{|z|=\epsilon/2} f(z) \sin(1/z) dz = Res_{z=0}(g).$$

It is left to compute the residue at 0: write the Taylor expansion of $f(z)$ around 0:

$$f(z)=\sum_{n=0}^{\infty}a_nz^n \Longrightarrow g(z)=\Big(\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!z^{2k+1}}\Big)\Big(\sum_{n=0}^{\infty}a_nz^n\Big),$$

so the coefficient of $1/z$ (and hence the residue) is given by

$$Res_{z=0}(g)=\sum_{k=0}^{\infty}\frac{(-1)^k a_{2k}}{(2k+1)!}.$$

So the statement is false and a counter example will be: any analytic $f(z)$ for which their Taylor coefficients are given by a_0, $a_2\neq 0$ and $a_{2k}=0$ for $k>1$, since we will get $Res_{z=0}(g)=a_0-a_2/6\neq a_0=f(0)$.

b)iii) Suppose $f(-z)=-f(z)$ for any z. As z_0 is an isolated singularity point of f, we can write its Laurent expansion at z_0:

$$f(z)=\sum_{m=1}^{\infty}\frac{b_m}{(z-z_0)^m}+a_0+\sum_{n=1}^{\infty}a_n(z-z_0)^n,$$

so in a neighborhood of $-z_0$ we get that the Laurent expansion of f at $z=z_0$ is

$$f(z)=-f(-z)=\sum_{m=1}^{\infty}\frac{(-1)^{m+1}b_m}{(z+z_0)^m}-a_0+\sum_{n=1}^{\infty}(-1)^{n+1}a_n(z+z_0)^n.$$

Therefore, $Res_{z=z_0}(f)=b_1=Res_{z=-z_0}(f)$.

If we know that $f(z)=f(-z)$, then similarly we get $Res_{z=z_0}(f)=-Res_{z=-z_0}(f)$.

b)iv) Suppose that $f(-z)=f(z)$ and that $0,\infty$ are isolated singularity point of f. Write the Laurent expansion of f at 0,

$$f(z)=\sum_{m=1}^{\infty}\frac{b_m}{z^m}+\sum_{n=0}^{\infty}a_nz^n,$$

so (by the uniqueness of the coefficients in the Laurent expansion)

$$f(z)=f(-z)=\sum_{m=1}^{\infty}\frac{(-1)^m b_m}{z^m}+\sum_{n=0}^{\infty}(-1)^n a_nz^n,$$

which implies that $b_m=0$ and $a_n=0$ for every $2\mid m,n$. In particular, $Res_{z=0}(f)=b_1=0$ and also $Res_{z=\infty}(f)=-a_1=0$.

c) Write

$$f(z)=\sum_{m=1}^{\infty}\frac{b_m}{z^m}+\sum_{n=0}^{\infty}a_nz^n,$$

so

$$f(cz)=\sum_{m=1}^{\infty}\frac{b_m}{c^mz^m}+\sum_{n=0}^{\infty}a_nc^nz^n$$

and thus $Res_{z=0}(f(cz))=\frac{b_1}{c}=\frac{1}{c}Res_{z=0}(f)$.

d)i) Let $f(z)=\frac{e^z}{\tan(z)}$. The (non removable) singularities of f are exactly the points z such that $\sin(z)=0$; inside the box we have exactly 3 of them: $z=-\pi,0,\pi$ and all of them are simple poles, with $Res_{z=z_0}(f) = Res_{z=z_0}(\frac{\cos(z)e^z}{\sin(z)}) = e^{z_0}$. By the residue theorem

$$\int_{\partial B} f(z)dz = 2\pi i(Res_{z=-\pi}(f) + Res_{z=0}(f) + Res_{z=\pi}(f)) = 2\pi i(e^{-\pi} + 1 + e^{\pi}).$$

d)ii) Let $f(z) = \frac{Log(z)}{\sin^3(z-2i)}$. Clearly in $Ball_1(2i)$ the function f is analytic except for at the singularity point $z = 2i$ which is a pole of order 3, so

$$Res_{z=2i}(f) = \frac{1}{2}\lim_{z\to 2i}\Big(\frac{(z-2i)^3 Log(z)}{\sin^3(z-2i)}\Big)^{(2)} = \dots$$

and by the residue theorem

$$\int_{|z-2i|=1} f(z)dz = 2\pi i Res_{z=2i}(f).$$

d)iii) The function $f(z) = \frac{\sin(1/z)}{z-1}$ has 2 singularity points inside $Ball_2(0)$ which are $z = 0, 1$. Therefore, the residue theorem tells us that

$$\int_{|z|=2} f(z)dz = 2\pi i(Res_{z=0}(f) + Res_{z=1}(f)).$$

Clearly $z = 1$ is a simple pole of f, thus

$$Res_{z=1}(f) = \lim_{z\to 1}(z-1)f(z) = \sin(1),$$

while $z = 0$ is an essential singularity of f, so we compute the residue by finding the Laurent expansion at $z = 0$ of the function and just taking the coefficient of $1/z$; here is the computation

$$\sin(\frac{1}{z}) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!z^{2n+1}},\ \frac{1}{z-1} = -\sum_{m=0}^{\infty} z^m$$

$$\implies f(z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{n+1}}{(2n+1)!}z^{m-2n-1},$$

therefore the coefficient of $1/z$ is given by (choosing $m = 2n$)

$$\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n+1)!} = -\sin(1);$$

so

$$\int_{|z|=2} f(z)dz = 2\pi i(\sin(1) - \sin(1)) = 0.$$

A shorter way to solve this is using the residue at ∞, as $z = 0, 1$ are the only singularities of f and they are inside $Ball_2(0)$, we have

$$\int_{|z|=2} f(z)dz = -2\pi i Res_{z=\infty}(f) = 2\pi i Res_{z=0}(\frac{1}{z^2}f(1/z)) = Res_{z=0}(\frac{\sin(z)}{z-z^2}) = 0.$$

d)iv) Let $f(z) = \frac{z}{e^{2\pi i z^2}-1}$. The singularities of f are exactly $z = \pm\sqrt{k}$ for every $k \in \mathbb{N}$, they are all simple poles, with

$$Res_{z=\pm\sqrt{k}}(f) = \frac{z}{4\pi i z e^{2\pi i z^2}}\ |_{z=\pm\sqrt{k}} = \frac{1}{4\pi i}.$$

Therefore, from the residue theorem we have

$$\int_{|z|=R} f(z)dz = 2\pi i \sum_{k:\sqrt{k}\le R} Res_{z=\pm\sqrt{k}}(f) = 2\pi i(2n+1)\frac{1}{4\pi i} = \frac{2n+1}{2}.$$

e) For every $|z|=1$ we know that $1/\overline{z} = z$, therefore

$$\int_\gamma e^{1/\overline{z}} + e^{-1/\overline{z}}dz = \int_\gamma e^z + e^{-z}dz = |_r^{-r} e^z - e^{-z} = 2(e^{-r} - e^r).$$

4. Question 4.

i) Let $f(x) = \frac{\cos^2(x)}{1-a\sin^2(x)}$. This is a symmetric function, therefore

$$\int_0^\pi f(x)dx = \frac{1}{2}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2}\int_0^{2\pi} f(x)dx.$$

Next, write

$$\cos(x) = \frac{e^{ix}+e^{-ix}}{2},\ \sin(x) = \frac{e^{ix}-e^{-ix}}{2i},\ z = e^{ix}:$$

$$\int_0^{2\pi} f(x)dx = \frac{1}{4}\int_{|z|=1} \frac{(z+1/z)^2}{1-\frac{a}{4}(z-1/z)^2}\frac{dz}{iz} = \frac{1}{i}\int_{|z|=1}\frac{(z^2+1)^2}{z(4z^2-a(z^2-1)^2)}dz.$$

As

$$4z^2 - a(z^2-1)^2 = (2z-\sqrt{a}(z^2-1))(2z+\sqrt{a}(z^2-1))$$

we get 4 points which are singularities- it remains the find these points and decide which of them lies inside $Ball_1(0)$ and finally calculate the residues at these points.

ii)

$$I := \int_{-\pi}^{\pi}\frac{\cos(nx)}{a-\cos(x)}dx = Re\Big(\int_{-\pi}^{\pi}\frac{e^{inx}}{a-\cos(x)}dx\Big) = Re\Big(\int_{|z|=1}\frac{z^n dz}{iz(a-\frac{1}{2}(z+\frac{1}{z}))}dz\Big)$$

$$= Re\Big(\frac{2}{i}\int_{|z|=1}\frac{z^n}{2az-z^2-1}dz\Big) = Re\Big(\frac{-2}{i}\int_{|z|=1}\frac{z^n}{(z-(a+\sqrt{a^2-1}))(z-(a-\sqrt{a^2-1}))}dz\Big)$$

as $a>1$, we know that $a+\sqrt{a^2-1}>1$ and that $0<a-\sqrt{a^2-1}<1$, therefore there is only 1 singularity (that is a simple pole) in $\{|z|<1\}$; by the residue theorem:

$$I = Re\Big(\frac{-2}{i}2\pi i Res_{z=a-\sqrt{a^2-1}}\Big(\frac{z^n}{2az-z^2-1}\Big)\Big)$$

$$= Re\Big(-4\pi\Big(\frac{z^n}{2a-2z}\Big)|_{z=a-\sqrt{a^2-1}}\Big) = -2\pi\frac{(a-\sqrt{a^2-1})^n}{\sqrt{a^2-1}}.$$

(iii)

$$I := \int_0^{2\pi}\frac{dt}{|ae^{it}-b|^4} = \int_0^{2\pi}\frac{dt}{(ae^{it}-b)^2(ae^{-it}-b)^2} = \int_0^{2\pi}\frac{e^{2it}dt}{(ae^{it}-b)^2(a-be^{it})^2}$$

$$= \int_{|z|=1}\frac{z}{(az-b)^2(a-bz)^2}\frac{dz}{i}$$

as $0<a<b$, among the 2 singularities $z=\frac{b}{a},\frac{a}{b}$, only $z=\frac{a}{b}$ is in $\{|z|<1\}$ and it is a pole of order 2, therefore by the residue theorem

$$I := 2\pi Res_{z=a/b}\Big(\frac{z}{(az-b)^2(a-bz)^2}\Big) = 2\pi\lim_{z\to a/b}\Big(\frac{z}{b^2(az-b)^2}\Big)^{(1)} = \ldots = \frac{2\pi(a^2+b^2)}{(b^2-a^2)^3}.$$