Selected Answers to HW 11

Remark See more solutions to integrals in Tirguls.

Question 1

Item a iii

The integral we need to compute is equal to

$$I = \int_0^\infty \frac{1}{1+x^{2n}} dx = \frac{1}{2} \int_{-\infty}^\infty \frac{1}{1+x^{2n}} dx.$$

To solve this integral from negative infinity to infinity, one can close the path by the upper semicircle. Then one has to compute the residues at all the singular points in the upper half-plane (just as we saw in class). Here's a different solution:

Denote by $\theta = \frac{\pi}{n}$ and consider the following three paths:

$$\gamma_R(t) = t, \ \gamma_R : [0, R] \to \mathbb{C}$$

$$\alpha_R(t) = Re^{it}, \ \gamma_R : [0, \theta] \to \mathbb{C}$$

$$\beta_R(t) = e^{i\theta}(R - t), \ \gamma_R : [0, R] \to \mathbb{C}.$$

We denote by Γ_R the concatenation of the three paths (the "slice" of angle θ). We calculate the integral

$$I = \int_0^\infty \frac{1}{1+x^{2n}} dx$$

by

$$I = \lim_{R\to\infty} \int_0^R \frac{1}{1+x^{2n}} dx = \lim_{R\to\infty} \int_{\Gamma_R} \frac{1}{1+z^{2n}} dz - \int_{\alpha_R} \frac{1}{1+z^{2n}} dz - \int_{\beta_R} \frac{1}{1+z^{2n}} dz$$

By the integral triangle inequality

$$\left| \int_{\alpha_R} \frac{1}{1+z^{2n}} dz \right| \leq 2\pi R \max_{|z|=R} \left| \frac{1}{1+z^{2n}} \right| \underset{R\to\infty}{\to} 0.$$

For β_R:

$$-\int_{\beta_R} \frac{1}{1+z^{2n}} dz = \int_0^R \frac{e^{i\theta}}{1+e^{i\theta 2n} t^{2n}} dt = \int_0^R \frac{e^{i\theta}}{1+t^{2n}} dt \underset{R\to\infty}{\to} e^{i\theta} I.$$

(A priori, the limit might not converge, how ever we know the improper integral exists from a comparison test). Thus we conclude:

$$(1 - e^{i\theta}) I = \lim_{R\to\infty} \int_{\Gamma_R} \frac{1}{1+z^{2n}} dz$$

Or

$$I = \frac{1}{1 - e^{i\theta}} \lim_{R\to\infty} \int_{\Gamma_R} \frac{1}{1+z^{2n}} dz.$$

By the Residue Theorem, the integral is equal to $2\pi i$ times the residues of the sigular points in the interior of Γ_R. The singular points of $\frac{1}{1+z^{2n}}$ are $e^{\frac{i(2\pi k+\pi)}{2n}}$ for all $k = 0, 1, ..., 2n-1$. In the interior of Γ_R we have only $e^{\frac{\pi}{2n}}$. Thus

$$I = \frac{2\pi i}{1-e^{i\theta}} Res(\frac{1}{1+z^{2n}}, e^{\frac{\pi}{2n}}).$$

By a theorem we saw in class, since $e^{\frac{\pi}{2n}}$ is a simple root, the residue is equal to 1 over the derivative of $1+z^{2n}$, i.e. $\frac{-1}{2ne^{\frac{-\pi}{2n}}} = \frac{-1}{2ne^{-\theta/2}}$. In conclusion

$$I = \frac{2\pi i}{1-e^{i\theta}} \frac{-1}{2ne^{-i\theta/2}} = \frac{\pi}{2n\sin(\frac{\pi}{2n})}.$$

item b i

Let $0 < \alpha < 1$ be some real number. Denote by

$$I = \int_{-\infty}^{\infty} \frac{e^{\alpha x}}{1+e^x} dx.$$

Consider the following four paths for some parameter R:

$$\gamma_1 : [-R, R] \to \mathbb{C},\ \gamma_1(t) = t,$$

$$\gamma_2 : [-, 2\pi] \to \mathbb{C},\ \gamma_2(t) = R + 2\pi i t,$$

$$\gamma_3 : [-R, R] \to \mathbb{C},\ \gamma_3(t) = 2\pi i - t,$$

$$\gamma_4 : [-R, R] \to \mathbb{C},\ \gamma_4(t) = -R + 2\pi i(1-t).$$

Let Γ_R be the concatenation of these four paths. Then

$$I = \lim_{R\to\infty} \int_{\Gamma_R} \frac{e^{\alpha z}}{1+e^z} dz - \sum_{j=2}^{4} \int_{\gamma_j} \frac{e^{\alpha} z}{1+e^z} dz.$$

By the Residue Theorem

$$\int_{\Gamma_R} \frac{e^{\alpha} z}{1+e^z} dz = 2\pi i \sum_{k=1}^{n} Res(\frac{e^{\alpha z}}{1+e^z}, z_k)$$

where z_k are the singular points of the function in the interior of Γ_R. The singular points of $\frac{e^{\alpha} z}{1+e^z}$ are when the denominator is 0. this occurs whenever $z = (\pi + 2\pi k)i$ for any integer k. In our case, the only singular point inside the closed path is πi. Since this is a simple root of the denominator, the residue is $\frac{e^{\alpha\pi i}}{e^{\pi i}} = -e^{\alpha\pi i}$.

Thus

$$\int_{\Gamma_R} \frac{e^{\alpha z}}{1+e^z} dz = -2\pi i e^{\alpha\pi i}.$$

For γ_3:

$$-\int_{\gamma_3} \frac{e^{\alpha} z}{1+e^z} dz = \int_{-R}^{R} \frac{e^{\alpha t} e^{\alpha 2\pi i}}{1+e^{2\pi i + t}} dt = e^{2\pi\alpha i} \int_{-R}^{R} \frac{e^{\alpha t}}{1+e^t} dt \underset{R\to\infty}{\to} e^{2\pi\alpha i} I.$$

(A priori, the limit might not converge, how ever we know the improper integral exists from a comparison test).

As for γ_2, γ_4, we get for $j = 2, 4$ that from the integral triangle inequality:

$$\left|\int_{\gamma_j} \frac{e^{\alpha z}}{1+e^z} dz\right| \le 2\pi \max_{Re(z)=\pm R} \left|\frac{e^{\alpha z}}{1+e^z}\right| \le 2\pi \max_{Re(z)=\pm R} \left|\frac{e^{\alpha Re(z)}}{\left|1-e^{Re(z)}\right|}\right| \underset{R\to\infty}{\to} 0.$$

In conclusion

$$I = -2\pi i e^{\alpha\pi i} + e^{2\pi\alpha i} I.$$

$$I = \frac{-2\pi i e^{\alpha\pi i}}{1-e^{2\pi\alpha i}} = \frac{\pi}{sin(\pi\alpha)}.$$

item b ii

For $a = 0$ this is by the previous item. For $a \neq 0$ just decompose $\frac{1}{s^2-a^2} = \frac{1}{2a}\left(\frac{1}{s-a} - \frac{1}{s+a}\right)$ and use the previous item as well. [Note that this could also be solved directly via complex integral as above.]

Question 3

item a

When a function is defined on some environment of infinity (i.e. on $R < |z|$ for some R), the residue at infinity is defined by

$$Res(f, \infty) = \frac{-1}{2\pi i}\int_{|z|=\rho} f(z)dz,$$

for some $\rho > R$.

We saw in class that the integral does not depend on the radius ρ.

In our case, since $\frac{p(z)}{q(z)}$ goes to 0 like $\frac{1}{\rho^2}$ (when rho goes to infinity), the limit of the expression

$$\left|\frac{-1}{2\pi i}\int_{|z|=\rho} f(z)dz\right| \underset{\rho\to infty}{\to} 0.$$

Thus the residue must be 0.

Another Solution Since $\lim_{z\to\infty}\frac{p(z)}{q(z)} = 0$ then $Res_{z=\infty}\frac{p(z)}{q(z)} = \lim_{z\to\infty} -z\frac{p(z)}{q(z)}$. Since the degree of $q(z)$ is greater than $p(z)$ by at least 2, then this limit also goes to 0. Thus the residue is 0.

item b

i. False. Take $f(z) = \frac{1}{z}$. On the one hand, we know that

$$Res(\frac{1}{z}, \infty) = -\frac{1}{2\pi i}\int_{|z|=R}\frac{dz}{z} = -1.$$

However, $Res(z, 0) = 0$.

ii. False. E.g. $e^{\frac{1}{z}}$ has a removable singularity at infinity but the residue is not 0 (it is $-c_{-1}$ from the series expansion of $e^{\frac{1}{z}}$ around 0).

iii. True. We saw in class that if $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$ then the residue at infinity is $-c_{-1}$. If f is analytic, then $c_{-1} = 0$.

item c

The integral we need to calculate is equal to $-2\pi i Res(f(z), \infty)$ for $f(z) = \frac{sin(1/z)}{cos(1/(z+i))}$. We saw in class that

$$Res(f(z), \infty) = -Res(z^{-2}f(\frac{1}{z}), 0).$$

Thus we need to calculate

$$Res(\frac{sin(z)}{z^2 cos(\frac{z}{iz+1})}, 0).$$

Denote

$$\frac{sin(z)}{z^2 cos(\frac{z}{iz+1})} = \frac{g(z)}{h(z)}$$

For $g(z) = \frac{sin(z)}{z}$ and $h(z) = zcos(\frac{z}{iz+1})$. Hence g is analytic at 0 and h has a simple 0, thus

$$Res(\frac{sin(z)}{z^2 cos(\frac{z}{iz+1})}, 0) = \frac{g(0)}{h'(0)} = \frac{1}{cos(1) + 0} = 1,$$

and the integral is equal to $2\pi i$.

item d

i. This is a simply connected domain and $f(z)$ is analytic there thus from a theorem we saw in class f has a primitive function there.

ii. No. Not all closed paths integrate to 0. Take a small circle around $z_0 = \frac{1}{\sqrt{2\pi}}$ - the integral is $2\pi i Res(f, z_0)$. The residue is not 0 (check at home).

iii. Yes. By Morera's theorem, we need to show that for every circle, the integral of $\int_{|z-a|=R} f(z)dz = 0$. Notice that if the circle does not circle the ball of radius 0 then it's interior is simply connected and thus the integral is 0. Now take some circle that circles 0. The integral is equal to equal to $-2\pi i Res(f(z), \infty)$.

The residue in this case is $Res(f(z), \infty) = Res(\frac{1}{z^2} f(\frac{1}{z}), 0) = Res(\frac{1}{z^2 sin(z^2)}, 0)$. The function $z^2 sin(z^2)$ is even, thus from previous homework its Laurent series is composed of even powers z^{2k}, and it has no residue at 0. Thus the integral is 0.

Question 4

item a

The residue at infinity is $Res(Log(\frac{z}{z+2}, \infty) = Res(\frac{1}{z^2} Log(\frac{1/z}{2+(1/z)}), 0)$. The function $g(z) = Log(\frac{1/z}{2+(1/z)}) = Log(\frac{1}{2z+1})$ is analytic around 0. Thus the residue of $\frac{1}{z^2}\frac{1}{z^2} Log(\frac{1/z}{2+(1/z)})$ is the coefficient multiplying z in the Taylor series of $g(z)$, i.e. $g'(0) = -1$. No. Assume towards contradiction that there was such a function. Then

$$\int_{|z|=10} f(z)dz = 0.$$

by the fundamental theorem for complex functions. On the other hand,

$$\int_{|z|=10} f(z)dz = -Res(f, \infty) = 1.$$

A contradiction.

item b

First we show that $g(z) = e^{f(z)/n}$ is an analytic branch of $\sqrt[n]{\frac{z}{z+2}})$. Indeed $g(z)^n = e^{f(z)} = e^{Log(\frac{z}{z+2}}$.

As $\frac{z}{z+2}$ is a non-positive real number if and only if $z \in [-2, 0]$ we can define $g(z)$ on $\mathbb{C} \setminus [-2, 0]$. Calculate the integral:

$$\int_{|z|=3} g(z)dz = -2\pi i Res(g, \infty).$$

Since $\lim_{z\to\infty} g(z) = e^{\frac{1}{n}\lim_{z\to\infty} f(z)} = e^0 = 1$, then

$$Res(g, \infty) = Res(z^{-2}(g(\frac{1}{z})), 0) = \frac{1}{z^2} e^{\frac{1}{n} Log(\frac{1}{2z+1})}.$$

The function $h(z) = e^{\frac{1}{n} Log(\frac{1}{2z+1})}$ is analytic around 0 thus the residue is it's first derivative at 0, $h'(0) = \frac{-1}{n}$(similar to our argument in item a). Thus the integral is equal to $\frac{2\pi i}{n}$.