HW12: SOLUTIONS

1. Question 1

a) Yes. As f is continuous, its image is path-connected. But the set $\mathbb{C}\setminus\{|z|=1\}$ is not path-connected, thus the image of f lies either in $\{|z|>1\}$ or in $\{|z|<1\}$. Both cases contradict the fact that the image of a non-constant entire function is a dense subset of $\mathbb{C}$.

Another solution: As $f \in \mathcal{O}(\mathbb{C})$, if f is bounded then by Liouville's theorem f is constant. If f is not bounded, then $|f(z)|$ is not bounded and is a real continuous function such that $|f(z)| \neq 1$ for all $z \in \mathbb{C}$, so we must have $|f(z)| > 1$ for all $z \in \mathbb{C}$. In particular it means that $f(z) \neq 0$ for all $z \in \mathbb{C}$ and hence the function $g(z) = 1/f(z)$ is entire and bounded, hence constant and so is f.

d)i) Let $f \in \mathcal{O}(\mathbb{C})$ be bounded and $a \neq b$ in $\mathbb{C}$. Define

$$g(z) = \frac{f(z)}{(z-a)(z-b)}.$$

For every $R > \max\{|a|,|b|\}$, we have by the residue theorem

$$\int_{|z|=R} g(z)dz = 2\pi i(Res_{z=a}(g)+Res_{z=b}(g)) = 2\pi i\Big(\frac{f(a)}{a-b}+\frac{f(b)}{b-a}\Big) = 2\pi i\Big(\frac{f(b)-f(a)}{b-a}\Big).$$

ii) f is bounded, say $|f(z)| \leq M$ for all $z \in \mathbb{C}$. For every $|z| = R$, we have

$$|g(z)| = \frac{|f(z)|}{|z-a||z-b|} \leq \frac{M}{|R-|a|||R-|b||}$$

and hence

$$\left|\int_{|z|=R} g(z)dz\right| \leq \frac{MR}{|(R-|a|)(R-|b|)|}\int_0^{2\pi} d\theta$$

which implies that

$$\lim_{R\to\infty}\int_{|z|=R} g(z)dz = 0.$$

As a corollary, we get from the first part of the question that

$$0 = 2\pi i\Big(\frac{f(b)-f(a)}{b-a}\Big) \Longrightarrow f(a) = f(b)$$

for every $a \neq b$ in $\mathbb{C}$, i.e., that f is constant.

2. Question 2

(i) Let $p(z) = z^5 + z + 1$. Notice that all the zeros of p are in $Ball_2(0)$: if $p(z_0) = 0$ and $|z_0| \geq 2$, then $-1 = z_0^5 + z_0$ implies

$$1 = |z_0^5 + z_0| = |z_0||1 + z_0^4| \geq 2||z_0^4| - 1| \geq 30$$

which is a contradiction. Therefore all the zeros of p satisfy $|z| < 2$. Thus by the residue theorem

$$\int_{|z|=2} \frac{dz}{p(z)} = 2\pi i Res_{z=\infty}\Big(\frac{1}{p(z)}\Big) = -2\pi i Res_{z=0}\Big(\frac{1}{z^2 p(1/z)}\Big) = -2\pi i Res\Big(\frac{z^3}{1+z^4+z^5}\Big) = 0,$$

since the last function is analytic at $z = 0$.

(ii) If R is big enough, all the zeros of $p(z)$ will be in $Ball_R(0)$ and thus by the residue theorem

$$\int_{|z|=R} \frac{dz}{p(z)} = 2\pi i Res_{z=\infty}\Big(\frac{1}{p(z)}\Big) = -2\pi i Res_{z=0}\Big(\frac{1}{z^2 p(1/z)}\Big) = -2\pi i Res\Big(\frac{z^{d-2}}{c_d + \ldots + c_0 z^d}\Big),$$

for $d \geq 2$ the function is analytic at $z = 0$ (since $c_d \neq 0$) and hence the integral is 0; for $d = 1$ we get a simple pole at $z = 0$ and hence

$$\int_{|z|=R} \frac{dz}{p(z)} = \frac{-2\pi i}{c_d};$$

finally, if $d = 0$, the integral is equal to 0 as $1/p(z)$ is a constant.

(v) Let

$$f(z) = \frac{1}{(e^{1-z}-1)(z^3-1)},$$

so

$$\int_{Re(z)=1/2} f(z)dz = \lim_{R\to\infty} \int_{\gamma_R} f(z)dz = \lim_{R\to\infty} \Big(\int_{\Gamma_R} f(z)dz - \int_{\eta_R} f(z)dz\Big),$$

where γ_R, Γ_R and η_R are given by

$$\gamma_R(x) = \frac{1}{2} + x,\ x \in [-R, R]$$

$$\eta_R(t) = \frac{1}{2} + Re^{it},\ t \in \Big[\frac{\pi}{2}, \frac{3\pi}{2}\Big]$$

and

$$\Gamma_R = \gamma_R \cup \eta_R.$$

First, let's show that the integral on η_R tends to 0: for every $z \in \eta_R$ we have $|z - 1/2| = R$ and hence $R - 1/2 \leq |z| \leq R + 1/2$ and also $Re(z) \leq 1/2$; therefore

$$|f(z)| \leq \frac{1}{||e^{1-z}| - 1||\,|z|^3 - 1|} \leq \frac{1}{|e^{1-Re(z)} - 1||(R-1/2)^3 - 1|} \leq \frac{1}{|\sqrt{e} - 1|}\frac{1}{(R-1/2)^3 - 1}$$

for R large enough and hence

$$\lim_{R\to\infty} \int_{\eta_R} f(z)dz = 0.$$

To compute the other integral we use the residue theorem, so we need to know what are the singularities of f: the function has singularities at the points

$$z_1 = e^{2\pi i/3}, z_2 = e^{-2\pi i/3}, 1, 1 + 2\pi n i,\ \forall n \in \mathbb{Z}$$

among all those points only the first two are inside Γ_R (notice that this is why we chose the left hand side of the circle around $1/2$ and not the right one) and they are both simple poles. So

$$\int_{\Gamma_R} f(z)dz = 2\pi i\Big(Res_{z=z_1}(f)+Res_{z=z_2}(f)\Big) = 2\pi i\Big(\frac{1}{(e^{1-z_1}-1)3z_1^2}+\frac{1}{(e^{1-z_2}-1)3z_2^2}\Big) = \dots$$

3. Question 3

Let $f(z) = \sin(\tan z) - \tan(\sin z)$; check what is the order of $z = 0$ as a zero of f: with some calculations one can write the Taylor polynomial of f of order 6 that is $\tan(z) = z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + o(z^6)$, then plug it together with the Taylor polynomial of $\sin(z)$ of order 6, to get

$$\begin{aligned} f(z) = \tan(z) - \frac{(\tan(z))^3}{3!} + \frac{(\tan(z))^5}{5!} - \sin(z) - \frac{(\sin(z))^3}{3} - \frac{2(\sin(z))^5}{15} + o(z^6) \\ = \dots = 0 + o(z^6), \end{aligned}$$

therefore

$$\lim_{z\to 0} \frac{f(z)}{z^6} = 0,$$

which means that

$$g(z) := \frac{f(z)}{z^4}$$

has a zero at $z = 0$ which is of order at least 2. Thus from a theorem studied in class, for every $0 < |\epsilon| \ll 1$ there are two different points such that $g(z) = \epsilon$, i.e., such that $f(z) = \epsilon z^4$. However, $z = 0$ is a solution of that equation and if $z_1 \neq 0$ is another solution so $-z_1$ is a third solution. So if there are 2 different solution to this equation, there mist be at least 3 different ones, as needed.

Notice that if you show— in some other way— that the order of $z = 0$ as a zero of $f(z)$ is equal to 7 (which is true) then automatically the statement holds, without the argument that if f has 2 solutions then it must have 3.

4. Question 4

b) Suppose $f, g \in \mathcal{O}(\mathbb{C})$ such that $f(z)^2 + g(z)^2 = 1$ for all $z \in \mathbb{C}$. Therefore

$$(f(z) + ig(z))(f(z) - ig(z)) = 1$$

for all $z \in \mathbb{C}$, so $f(z) + ig(z) \neq 0$ for all $z \in \mathbb{C}$. Therefore, from the first part of the question (since $f + ig \in \mathcal{O}(\mathbb{C})$), there exists $h \in \mathcal{O}(\mathbb{C})$ such that

$$e^{h(z)} = f(z) + ig(z) \Longrightarrow f(z) - ig(z) = \frac{1}{f(z) + ig(z)} = e^{-h(z)},$$

so

$$f(z) = \frac{1}{2}(e^{h(z)} + e^{-h(z)}),\ g(z) = \frac{1}{2i}(e^{h(z)} - e^{-h(z)}).$$

Let $\phi(z) := -ih(z) \in \mathcal{O}(\mathbb{C})$, so we get that

$$f(z) = \cos(\phi(z)),\ g(z) = \sin(\phi(z)).$$

The other direction is direct since $\sin^2(z) + \cos^2(z) = 1$ in $\mathbb{C}$.

c) Let f be meromorphic in $\mathbb{C}$ and suppose that $Im(f) \cap \mathbb{R}_{\leq 0} = \emptyset$, where $Im(f)$

is the image of f, in particular we get that $f(z) \neq 0$ for any z in the domain of f. Define

$$g(z) = 1 + \frac{1}{f(z)},$$

as f is meromorphic we get that all the singularities of f are now removable points of g and moreover they are zeros of g; therefore $g \in \mathcal{O}(\mathbb{C})$ and also

$$Im(g) \cap \mathbb{R}_{\leq 0} = \emptyset.$$

Notice that $1/f(z)$ might vanish at some points (which are singularities of f) and that is why we added 1. We actually changed the question to be about entire function instead of meromorphic, but now the answer is obvious:

We get that $\log g(z)$ entire but $\log g(z)$ has a bounded imaginary part, so using Liouville's theorem we can get that the imaginary part of $\log g$ is constant and so is $\log g$ (due to the Cauchy–Riemann equations for example) and g as well. As g is constant, it is easy to see that f must be a constant.

5. Question 5

a) For R large enough, all the zeros of p are inside $Ball_R(0)$, therefore by the argument of principle

$$\int_{|z|=R} \frac{p'(z)}{p(z)} dz = 2\pi i(N - P),$$

where N stands for the number of zeros of $p(z)$ in $Ball_R(0)$ and P stands for the number of poles of $p(z)$ in $Ball_R(0)$, including multiplicities. Clearly $P = 0$ and $N = n$, therefore

$$\int_{|z|=R} \frac{p'(z)}{p(z)} dz = 2\pi i n.$$

f) Let

$$p(z) = z^4 + 8z^3 + 3z^2 + 8z + 3.$$

First observation is that $p(z) \neq 0$ for all $z \in i\mathbb{R}$: if $z = iy \in i\mathbb{R}$ (with $y \in \mathbb{R}$), then $p(z) = 3 - 3y^2 + y^4 + i(8y - 8y^3) \neq 0$, since the real part, which is $3 - 3y^2 + y^4$, never vanishes for $y \in \mathbb{R}$. Next, for any $R > 0$ let $\Gamma_R = \eta_R \cup \gamma_R$, where

$$\gamma_R(t) = Re^{it}, t \in [-\pi/2, \pi/2]$$

and

$$\eta_R(x) = -ix,\ x \in [-R, R].$$

As $p(z) \neq 0$ for any z on η_R and also on γ_R for large enough R (more precisely we need $R > \max\{|z_1|, \ldots, |z_4|\}$ where $z_1, \ldots, z_4$ are the zeros of $p(z)$), we can use the argument of principle to conclude that

$$\int_{\Gamma_R} \frac{p'(z)}{p(z)} dz = 2\pi i N_R,$$

where N_R is the number of zeros of $p(z)$ inside Γ_R; therefore the number of zeros of $p(z)$ in $\{z : Re(z) > 0\}$ is equal to

$$\lim_{R\to\infty} N_R = \frac{1}{2\pi i} \lim_{R\to\infty} \Big(\int_{\gamma_R} \frac{p'(z)}{p(z)} dz + \int_{\eta_R} \frac{p'(z)}{p(z)} dz \Big).$$

A straightforward computation shows that if $z = iy$, then $Re(p(iy)) = 3-3y^2+y^4 > 0$, thus $\log(p(z))$ is defined in $i\mathbb{R}$, however notice that $p(iR) = R^4(1+O(1/R))$ and $p(-iR) = R^4(1+O(1/R))$, so

$$\lim_{R\to\infty}\int_{\eta_R}\frac{p'(z)}{p(z)}dz = \lim_{R\to\infty}\big[\log(p(-iR)) - \log(p(iR))\big] = \lim_{R\to\infty}\log(1+O(1/R)) = 0.$$

Similarly, as $z \to \infty$, we have

$$\frac{p'(z)}{p(z)} = \frac{4}{z} - O(1/z^2),$$

which implies that

$$\lim_{R\to\infty}\int_{\gamma}\frac{p'(z)}{p(z)}dz = \lim_{R\to\infty}\Big[\int_{\gamma}\frac{4dz}{z} + \int_{\gamma}O(1/z^2)dz\Big] = 4\pi i.$$

Finally, we get that

$$\lim_{R\to\infty}N_R = \frac{1}{2\pi i}4\pi i = 2.$$