Selected Answers to HW 13

Question 1

Item a

Let $f : U \to \mathbb{C}$ be a non-constant function. We showed in class that for every inner point $z_0 \in U$ where $f'(z_0) \neq 0$ there is some $\varepsilon > 0$ so that $f(Ball_\varepsilon(z_0))$ is open. We now show this for any arbitrary z_0 without the assumption that $f'(z_0) \neq 0$.

Let z_0 be so that $f'(z_0) = 0$ and without loss of generality suppose also that $f(z_0) = 0$ (why?). Thus f has $ord_{z_0}(f) = k \geq 2$. We will use what we prove in part c of this question: that there is an analytic function $h(z) = (z - z_0)g(z)$ so that $h(z)^k = f(z)$, and so that $g(z)$ is analytic and $g(z_0) \neq 0$. As $h'(z_0) = g(z_0) \neq 0$ then by the open-function function theorem, there is some $Ball_{\varepsilon'}(z_0)$ so that $h(Ball_\varepsilon(z_0))$ is open.

Additionally, the function $z \mapsto z^k$ also sends open sets to open sets (for open sets that don't contain 0 this is true by the open function theorem we saw in class, and for balls around 0 this is true since it sends $Ball_\varepsilon(0)$ to $Ball_{\varepsilon^k}(0)$). Thus $h^k(Ball_\varepsilon(0)$ is also open as needed.

The claim in question does not hold for any arbitrary $f : U \to \mathbb{R}^2$ even if it is C^∞. For example, take $f : \mathbb{R}^2 \to \mathbb{R}^2$ that is defined by $f(x, y) = (x^2, y^2)$. It sends the open set $\mathbb{R}^2$ to $[0, \infty) \times [0, \infty)$ which is not open.

Item b

No. We saw in the previous item that the image of f must be an open set. $\mathbb{C} \setminus (0, 1)$ is not an open set (e.g. because 1 is not an inner point).

Item c

Indeed, let $f(z) = (z - z_0)^n \phi(z)$ be some function with a 0 of order n at z_0 only. We already saw in the previous homework that we can define $g(z) = (z - z_0)h(z)$ where $g'(z_0) = h(z_0) \neq 0$ and $f(z) = (g(z))^n$ in *some neighbourhood around* z_0. We now prove that we can in fact define $h(z)$ in all U.

We write $h(z) = \sqrt[n]{\phi(z)}$, and we can define h whenever the domain has no loop around 0. Assume towards contradiction that there is there is some closed path $\gamma \subset U$ around 0. On the one hand,

$$\eta(\phi \circ \gamma, 0) = \int_{\phi \circ \gamma} \frac{1}{z} dz \neq 0.$$

On the other hand,

$$\int_{\phi \circ \gamma} \frac{1}{z} dz = \int_\gamma \frac{\phi'(z)}{\phi(z)} = 2\pi i(N - P)$$

from the argument principle. As ϕ has no zeros nor poles in U then this integral is 0 - a contradiction.

Question 2

Item b

We can see how many times the path $p(\partial Ball_1(0))$ by the winding number:

$$\eta(p(\partial Ball_1(0)), 0) = \frac{1}{2\pi i} \int_{p(\partial Ball_1(0))} \frac{1}{z} dz.$$

This is equal to

$$\int_{p(\partial Ball_1(0))} \frac{1}{z} dz = \int_{\partial Ball_1(0)} \frac{p'(z)}{p(z)} dz,$$

and by the argument principle this is equal to the number of zeros minus the number of poles in the unit ball (note that $p(z)$ has no zero's in $\partial Ball_1(0)$ thus we can indeed use the argument principle). $p(z)$ has no poles so

$$\eta(p(\partial Ball_1(0)), 0) = N_{Ball_1(0)}.$$

Note that on the boundary $\partial Ball_1(0) = \{z : |z| = 1\}$

$$|3z^3| \geq |z^{10} + 1|$$

Thus by Roche's Lemma, the number of zeros that $p(z) = (3z^3) + (z^{10} + 1)$ has in the ball is equal to the number of zeros that $3z^3$ has in the ball - i.e. 3 zeros.

Thus we conclude that $p(\partial Ball_1(0))$ circles 0 three times.

Item c

Denote $f(z) = \frac{p(z)}{q(z)}$. Note that $\bar{f}(\bar{z})$ is also analytic (why?), which implies that $\bar{f}(1/\bar{z})$ is also analytic. But

$$\bar{f}(1/\bar{z}) = \bar{f}(z) = f(z)$$

(initially this is true on the boundary of the ball since there f is real, but from the uniqueness theorem it is true in all $\mathbb{C}$). Thus in particular, if $f(z) = 0$ then $f(1/\bar{z}) = 0$ (and same for poles).

Now if $p(z) \neq 0$ for any $|z| = 1$, we can say a bit more:

For $\gamma : [0, 2\pi] \to \mathbb{C}$, $\gamma = e^{it}$ the canonical parametrization for the unit circle, $f \circ \gamma(t) \in \mathbb{R}$. Thus the winding number of 0

$$\eta(f \circ \gamma, 0) = \int_{f \circ \gamma} \frac{1}{z} dz = 0.$$

Thus by the argument principle in $Ball_1(0)$ the number of zeros and the number of poles is equal.

Item e

See solution in last Tirgul.

Question 3

Item b

As seen before, when $ord_{z_0}(f) = p$ we can write $f(z) = [(z - z_0)h(z)]^p$, where h is some analytic function and $h(z_0) \neq 0$ in some local neighbourhood around z_0.

The inner function $g(z) = (z - z_0)h(z)$ has a derivative $g'(z_0) = h(z_0) \neq 0$ thus is conformal at z_0.

Hence we can change to local coordinates and take $f(z) = z^p$. For any γ_1, γ_2 who intersect at z_0, and have angle

$$\angle(\gamma_1, \gamma_2) = \alpha$$

Then

$$\angle(\gamma_1(t)^p, \gamma_2(t)^p) = p\alpha.$$

Indeed, when $h(z) = z^p$ then $h(re^{i\theta}) = r^p e^{i(p\theta)}$ the function $z \mapsto z^p$ multiplies angles by a factor of p and thus

$$\angle(\gamma_1(t)^p, \gamma_2(t)^p) = p\alpha.$$

Question 4

Item c

TL;DR - take $T_a \circ S_{b-ad} \circ Inv \circ T_d$ (and check at home that this works).

Longer explanation (or - how we found these specific transformations):

Without loss of generality we can assume that $c = 1$ (why?). First we would like to get $z + d$ in the denominator, so we take the following elementary transformations:

$$Inv \circ T_d(z) = \frac{1}{z+d}$$

. Now we note that for any constant K

$$K + \frac{1}{z+d} = \frac{Kz + Kd}{z+d},$$

So at the end we would like to have apply T_a. If we would apply T_a directly we would get

$$T_a \circ Inv \circ T_d(z) = \frac{az + ad}{z+d},$$

and to fix this we apply S_{b-ad} before T_a and get

$$T_a \circ S_{ad-b} \circ Inv \circ T_d(z) = \frac{az + b}{z+d}.$$

As we assume that $ad - bc = ad - b \neq 0$ these are all elementary transformations.

Item e

Note that if $c = 0$ then f is a linear transformation and then the proof is direct (check at home).

Let's prove for the case where $c \neq 0$. From item c, we can write

$$f(z) = T_a \circ S_{b-ad} \circ Inv \circ T_d$$

where T_a, S_{b-ad}, T_d all send circles/lines to circles/lines, so it is enough to show that $g(z) = \frac{1}{z}$ sends circles/lines to circles/lines.

A line or circle in $\mathbb{R}^2$ are presented by $\tilde{a}(x^2 + y^2) + \tilde{b}x + \tilde{c}y + \tilde{d} = 0$ (if $\tilde{a} = 0$ then a line, otherwise a circle). Furthermore, notice that for any $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \in \mathbb{R}$ this equation defines a line, a circle, a point (e.g. $x^2 + y^2 = 0$) or the empty set (e.g. $x^2 + y^2 + 1 = 0$.

Thus in $\mathbb{C}$ we have a line or circle represented as

$$az\bar{z} + bz + c\bar{z} + d = 0.$$

By dividing by $\bar{z}z$ we see that this equation holds if and only if for $w = f(z) = \frac{1}{z}$:

$$a + b\bar{w} + cw + dw\bar{w} = 0.$$

This is not a point or the empty set, since the original equation has infinitely many satisfying points, that also satisfy the new equation in w. Thus we get either a line or a circle.