Selected Answers to HW 2

HW 2

Question 2

Item a

i The proof here is the same as the proof in Calculus I. We need to show that

$$\lim_{z \to z_0} \frac{f(z)g(z) - f(z_0)g(z_0)}{z - z_0} = f(z_0)g'(z_0) + f'(z_0)g(z_0).$$

Indeed:

$$\lim_{z \to z_0} \frac{f(z)g(z) - f(z_0)g(z_0)}{z - z_0} =$$

$$\lim_{z \to z_0} \frac{f(z)g(z) - f(z_0)g(z) + f(z_0)g(z) - f(z_0)g(z_0)}{z - z_0} =$$

$$\lim_{z \to z_0} f(z_0)\frac{g(z) - g(z_0)}{z - z_0} + \frac{f(z) - f(z_0)}{z - z_0}g(z).$$

From limit arithmetics rules and the fact that

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0), \ \lim_{z \to z_0} \frac{g(z) - g(z_0)}{z - z_0} = g'(z_0), \ \lim_{z \to z_0} g(z) = g(z_0),$$

we get that the limit above is equal to $f(z_0)g'(z_0) + f'(z_0)g(z_0)$.

ii First we calculate the derivative of $h(z)$. Later on in the course we'll do so immediately, but for now let's see how to do it from the definition (just as in the real case). Define $h(z) = \frac{1}{z}, h : \mathbb{C} \setminus \{0\} \to \mathbb{C}$. We show by definition that this function is differentible and it's derivative is $\frac{-1}{z^2}$.

$$\lim_{z \to z_0} \frac{h(z) - h(z_0)}{z - z_0} - \frac{-1}{z_0^2} = \lim_{z \to z_0} \frac{1}{z - z_0} \left(\frac{1}{z} - \frac{1}{z_0} + \frac{z - z_0}{z_0^2} \right)$$

$$= \lim_{z \to z_0} \frac{1}{z - z_0} \left(\frac{z_0^2 - z_0 z + z^2 - z_0 z}{z_0^2 z} \right) =$$

$$\lim_{z \to z_0} \frac{z - z_0}{z_0^2 z} = 0.$$

Now to calculate the derivative of $\frac{f(z)}{g(z)}$ we can just use the multiplication rule we proved previously and the chain rule we'll prove below to get that:

$$\left(\frac{f(z)}{g(z)} \right)' = \frac{f'(z)g(z) - g'(z)f(z)}{g^2(z)}.$$

Item b

The composed function We are familiar with the chain rule for functions $f : \mathbb{R}^2 \to \mathbb{R}^2$. Denote by

$$f = u + iv, g = \hat{u} + i\hat{v}$$

and

$$f(g(x+iy)) = u(\hat{u}, \hat{v}) + iv(\hat{u}, \hat{v}).$$

then we represent Jacobian matrix by

$$f'(g(x+iy)) = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$

$$g'(x+iy) = \begin{bmatrix} \hat{u}_x & \hat{u}_y \\ \hat{v}_x & \hat{v}_y \end{bmatrix}.$$

Then $f(g(z))$ is differentiable (as a function to $\mathbb{R}^2$ and its differential is the multiplication of these matrices, i.e.

$$(f(g(z))' = \begin{bmatrix} u_x\hat{u}_x + u_y\hat{v}_x & u_y\hat{u}_x + v_y\hat{u}_y \\ u_y\hat{v}_x + v_x\hat{v}_y & u_y\hat{v}_x + v_y\hat{v}_y \end{bmatrix}.$$

By C-R equations for f, g we can change $u_x = v_y, u_y = -v_x$ (and the same for $\hat{u}, \hat{v}$) and get:

$$(f(g(z))' = \begin{bmatrix} u_x\hat{u}_x - u_y\hat{u}_y & u_y\hat{u}_x + u_x\hat{u}_x \\ -u_y\hat{u}_y - u_y\hat{u}_x & -u_y\hat{u}_y + u_x\hat{u}_x \end{bmatrix}.$$

The four coordinates are the derivatives of the composed real and imaginary parts of $f(g(z))$. As we can see the composed function's partial derivatives also satisfy C-R equations. From the theorem we saw in class, it is also differentiable in the complex sense.

Direct calculations show that $(f(g(z))' = f'(g(z))g'(z)$ (check at home that what we got in the matrix matches what we get when we multiply directly).

Item c

i Let

$$f(z) = \frac{ix+1}{y} = u(x,y) + iv(x,y),$$

where $u(x,y) = \frac{1}{y}$ and $v(x,y) = \frac{x}{y}$. Then

$$u_x(x,y) = 0 = v_y(x,y) = \frac{-x}{y^2} \iff x = 0$$

$$u_y(x,y) = \frac{-1}{y^2} = -v_x(x,y) = \frac{-1}{y} \iff y = y^2 \iff y = 0, 1$$

so the only point in the domain of f for which the C-R equations hold is $x = 0, y = 1$ and it is $\mathbb{C}-$differentiable as u, v are differentiable at $(0,1) \Longrightarrow f$ is $\mathbb{C}-$differentiable in $\{i\}$.

ii One can use the C-R equations for f, but we can use another trick: if f is $\mathbb{C}-$differentiable at $z_0 \neq 0$, then using arithmetics we know that $\frac{f(z)}{z} = Re(z)$ is $\mathbb{C}-$differentiable at z_0, hence the C-R equations hold:

$$Re(z) = x + i0 \Longrightarrow u_x = 1 = v_y = 0$$

and that is a contradiction; So f is not $\mathbb{C}-$differentiable at $z_0 \neq 0$. However,

$$\lim_{z\to 0} \frac{f(z) - f(0)}{z} = \lim_{z\to 0} Re(z) = 0$$

so f is $\mathbb{C}-$differentiable at 0.

iii Let $f(z) = u(x,y) + iv(x,y)$ where

$$u(x,y) = \cos(x)\cosh(y), v(x,y) = -\sin(x)\sinh(y).$$

Recall that $(\cosh(x))' = \sinh(x)$ and $(\sinh(x))' = \cosh(x)$, then

$$u_x = -\sin(x)\cosh(y) = v_y,\ u_y = \cos(x)\sinh(y) = -v_x,$$

so the C-R equations hold for all points in $\mathbb{C}$. Moreover, the functions u, v are differentiable in $\mathbb{R}^2$ and therefore f is $\mathbb{C}-$differentiable (hence analytic) in $\mathbb{C}$.

Item d

We write $f(x+iy) = u(x,y) + iv(x,y)$. Then $x = rcos(\phi), y = rsin(\phi)$. Hence

$$f(r,\phi) = u(rcos(\phi), rsin(\phi)) + iv(rcos(\phi), rsin(\phi)).$$

We differentiate by r and by ϕ using the chain rule:

$$\frac{\partial f}{dr} = \frac{\partial f}{dx}\frac{\partial x}{dr} + \frac{\partial f}{dy}\frac{\partial y}{dr},$$

$$\frac{\partial f}{d\phi} = \frac{\partial f}{dx}\frac{\partial x}{d\phi} + \frac{\partial f}{dy}\frac{\partial y}{d\phi}.$$

Indeed

$$\frac{\partial f}{dr} = (u_x + iv_x)cos(\phi) + (u_y + iv_y)sin(\phi).$$

$$\frac{\partial f}{d\phi} = (u_x + iv_x)(-rsin(\phi)) + (u_y + iv_y)rcos(\phi).$$

$$ir\frac{\partial f}{dr} = (iu_x - v_x)rcos(\phi) + (iu_y - v_y)rsin(\phi).$$

We rearrange $ir\frac{\partial f}{dr}$ using $C - R$ and get

$$ir\frac{\partial f}{dr} = (u_x + iv_x)(-rsin(\phi)) + (u_y + iv_y)rcos(\phi) = \frac{\partial f}{d\phi}.$$

Question 3

Recall that $e^{x+iy} = e^x\big(\cos(x) + i\sin(x)\big)$.

item a

Let us write $1+i$ in its polar coordinates:

$$r = \sqrt{1+1} = \sqrt{2}, \theta = \arctan(1) = \frac{\pi}{4} + \pi k,\ k \in \mathbb{Z} \Longrightarrow 1 + i = \sqrt{2}e^{i\pi/4}$$

Write $z = x + iy$, then

$$e^z = 1 + i \iff e^x e^{iy} = \sqrt{2}e^{i\pi/4} \iff \frac{e^x}{\sqrt{2}}e^{i(y-\pi/4)} = 1$$
$$\iff \frac{e^x}{\sqrt{2}} = 1, y - \frac{\pi}{4} = 2\pi k,\ k \in \mathbb{Z} \iff x = \ln\sqrt{2}, y = \frac{\pi}{4} + 2\pi k,\ k \in \mathbb{Z}$$
$$\iff z = \ln\sqrt{2} + i(\frac{\pi}{4} + 2\pi k),\ k \in \mathbb{Z}.$$

item b

i Let $z = x_z + iy_z$ and $w = x_w + iy_w$. Then

$$e^{z+w} = e^{(x_z+x_w)+i(y_z+y_w)} = e^{x_z+x_w}(\cos(y_z + y_w) + i\sin(y_z + y_w)) =$$

recall that

$$\cos(y_z + y_w) = \cos(y_z)\cos(y_w) - \sin(y_z)\sin(y_w)$$

and

$$\sin(y_z + y_w) = \sin(y_z)\cos(y_w) + \cos(y_z)\sin(y_w),$$

so

$$\begin{aligned} e^{z+w} &= e^{x_z}e^{x_w}\big(\cos(y_z)\cos(y_w) - \sin(y_z)\sin(y_w) + i(\sin(y_z)\cos(y_w) + \cos(y_z)\sin(y_w))\big) \\ &= e^{x_z}\big(\cos(y_z) + i\sin(y_z)\big)e^{x_w}\big(\cos(y_w) + i\sin(y_w)\big) = e^z e^w. \end{aligned}$$

ii Using part **i)** and induction, or alternatively using de Moivre:

$$\begin{aligned} (e^z)^n &= \big(e^x(\cos(y) + i\sin(y))\big)^n = e^{xn}\big(\cos(y) + i\sin(y)\big)^n \\ &= e^{xn}\big(\cos(ny) + i\sin(ny)\big) = e^{xn+iyn} = e^{nz}. \end{aligned}$$

iii We have

$$\overline{e^z} = \overline{e^x(\cos(y) + i\sin(y))} = e^x(\cos(y) - i\sin(y)) = e^x(\cos(-y) + i\sin(-y)) = e^{x-iy} = e^{\overline{z}}.$$

iv We have

$$|e^z| = |e^x| \cdot |\cos(y) + i\sin(y)| = e^x.$$

v We have

$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z(\cos(2\pi) + i\sin(2\pi)) = e^z.$$

item d

If $x \in 2\pi\mathbb{Z}$, then $e^{ix} = 1$ and hence the sum is equal $n+1$. Let $x \in \mathbb{R} \setminus \{2\pi\mathbb{Z}\}$, then $e^{ix} \neq 1$ and hence

$$\begin{aligned} \sum_{k=0}^{n} e^{ikx} = \sum_{k=0}^{n}(e^{ix})^k &= \frac{1-(e^{ix})^{n+1}}{1-e^{ix}} = \frac{1-e^{ix(n+1)}}{1-e^{ix}} = \frac{(1-e^{ix(n+1)})(1-e^{-ix})}{(1-e^{ix})(1-e^{-ix})} \\ &= \frac{1-e^{-ix}-e^{ix(n+1)}+e^{ixn}}{(1-\cos(x))^2+\sin^2(x)} = \frac{1-e^{-ix}-e^{ix(n+1)}+e^{ixn}}{2-2\cos(x)} \end{aligned} \tag{0.1}$$

which implies that

$$\left|\sum_{k=0}^{n} e^{ikx}\right| = \left|\frac{1-e^{ix(n+1)}}{1-e^{ix}}\right| \leq \frac{2}{\sqrt{4\sin^2\left(\frac{x}{2}\right)}} = \frac{1}{\sin\left(\frac{x}{2}\right)}.$$

item e

i Using (0.1), we have

$$\sum_{k=0}^{n-1}\cos(a+kb) = Re\left(\sum_{k=0}^{n-1} e^{i(a+kb)}\right) = Re\left(e^{ia}\sum_{k=0}^{n-1}(e^{ib})^k\right) = Re\left(e^{ia}\frac{1-e^{-ib}-e^{ibn}+e^{ib(n-1)}}{2-2\cos(b)}\right)$$
$$= \frac{\cos(a)(1-\cos(b)-\cos(bn)+\cos(b(n-1)))-\sin(a)(\sin(b)-\sin(bn)+\sin(b(n-1)))}{2-2\cos(b)}.$$

ii

$$\sum_{k=0}^{n}\binom{n}{k}\cos(a+kb) = \sum_{k=0}^{n}\binom{n}{k}Re\left(e^{i(a+kb)}\right) = Re\left(e^{ia}\sum_{k=0}^{n}\binom{n}{k}(e^{ib})^k\right) = Re\left(e^{ia}(e^{ib}+1)^n\right)$$
$$= Re\left(e^{ia}(e^{i\frac{b}{2}})^n(e^{i\frac{b}{2}}+e^{-i\frac{b}{2}})^n\right) = Re\left(e^{i(a+\frac{bn}{2})}2^n\cos^n\left(\frac{b}{2}\right)\right) = 2^n\cos^n\left(\frac{b}{2}\right)\cos\left(a+\frac{bn}{2}\right)$$

iii If $t \in 2\pi\mathbb{Z}$, then $e^{it} = 1$ and hence the limit is equal to 1. Otherwise, $e^{it} \neq 1$ implies, using (0.1), that

$$\left|\frac{1+e^{it}+\ldots+e^{nit}}{n}\right| = \left|\frac{1-e^{-it}-e^{it(n+1)}+e^{itn}}{(2-2\cos(t))n}\right| \leq \frac{4}{(2-2\cos(t))n}$$

thus by the sandwich rule, our limit is equal to 0.

item f

Let $f(z) = e^z = u(x,y)+iv(x,y)$, where

$$u(x,y) = e^x\cos(y),\ v(x,y) = e^x\sin(y).$$

Thus we have the C-R equations

$$u_x = e^x\cos(y) = v_y,\ u_y = -e^x\sin(y) = -v_x$$

for every $(x,y)\in\mathbb{R}^2$ and clearly u,v are differentiable in $\mathbb{R}^2$, hence f is $\mathbb{C}-$differentiable in $\mathbb{C}$. Moreover, for every $z = x+iy$, we have

$$f'(z) = u_x(x,y)+iv_x(x,y) = e^x\cos(y)+ie^x\sin(y) = e^x\big(\cos(y)+i\sin(y)\big) = e^z.$$

item g

Let $0\neq w = re^{i\theta}\in\mathbb{C}$ and $z\in\mathbb{C}$; Write $\frac{1}{z} = x+iy$, thus

$$e^{\frac{1}{z}} = w \iff e^x = r, y = \theta+2\pi n, n\in\mathbb{Z} \iff x = \ln(r), y = \theta+2\pi n,\ n\in\mathbb{Z}$$

If we denote

$$z_n = \frac{1}{\ln(r)+i(\theta+2\pi n)},$$

then $f(z_n) = w$ for every $n\in\mathbb{Z}$ while $\left|\frac{1}{z_n}\right| = \ln^2(r)+(\theta+2\pi n)^2 \to\infty$ as $n\to\infty$, meaning that for $n\in\mathbb{N}$ large enough, we have $\left|\frac{1}{z_n}\right| > \frac{1}{\epsilon}$ hence $|z_n| < \epsilon$. This shows that the mapping $e^{\frac{1}{z}} : Ball_\epsilon(0)\setminus\{0\}\to\mathbb{C}\setminus\{0\}$ is onto and every $w\in\mathbb{C}\setminus\{0\}$ is achieved infinitely many times. Moreover, the sequence z_n we built above is satisfying $z_n\to 0$, while $f(z_n) = w$, so clearly the limit $\lim_{z\to 0} e^{\frac{1}{z}}$ does not exist!

Question 4

Define

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}.$$

item a

Let

$$f(z) = \cos(z) = \frac{1}{2}(e^{ix-y} + e^{-ix+y}) = \frac{1}{2}\Big(e^{-y}\big(\cos(x) + i\sin(x)\big) + e^{y}\big(\cos(x) - i\sin(x)\big)\Big)$$
$$= u(x,y) + iv(x,y)$$

and

$$g(z) = \sin(z) = \frac{1}{2i}(e^{ix-y} - e^{-ix+y}) = \frac{-i}{2}\Big(e^{-y}\big(\cos(x) + i\sin(x)\big) - e^{y}\big(\cos(x) - i\sin(x)\big)\Big)$$
$$= \widetilde{u}(x,y) + i\widetilde{v}(x,y)$$

where

$$u(x,y) = \frac{1}{2}(e^{-y} + e^{y})\cos(x) = \cos(x)\cosh(y), \quad v(x,y) = \frac{1}{2}(e^{-y} - e^{y})\sin(x) = -\sin(x)\sinh(y)$$

and

$$\widetilde{u}(x,y) = \frac{1}{2}(e^{-y} + e^{y})\sin(x) = \sin(x)\cosh(y), \quad \widetilde{v}(x,y) = \frac{1}{2}(e^{y} - e^{-y})\cos(x) = \cos(x)\sinh(y)$$

Check the C-R equations for f:

$$u_x = -\frac{1}{2}(e^{-y} + e^{y})\sin(x) = v_y = -\widetilde{u}, \quad u_y = \frac{1}{2}(e^{y} - e^{-y})\cos(x) = -v_x = \widetilde{v}$$

they hold everywhere in $\mathbb{R}^2$ and as u, v are differentiable in $\mathbb{R}^2$ (they have continuous derivatives) then f is analytic in $\mathbb{C}$ and we have

$$f'(z) = u_x(x,y) + iv_x(x,y) = -\widetilde{u}(x,y) - i\widetilde{v}(x,y) = -g(z) = \sin(z)$$

Similarly, the C-R equations for g are

$$\widetilde{u}_x = \frac{1}{2}(e^{-y} + e^{y})\cos(x) = \widetilde{v}_y = u, \quad \widetilde{u}_y = \frac{1}{2}(e^{y} - e^{-y})\sin(x) = -\widetilde{v}_x = -v$$

they hold everywhere in $\mathbb{R}^2$ and as $\widetilde{u}, \widetilde{v}$ are differentiable in $\mathbb{R}^2$ (they have continuous derivatives) then g is analytic in $\mathbb{C}$ and we have

$$g'(z) = \widetilde{u}_x(x,y) + i\widetilde{v}_x(x,y) = u(x,y) + iv(x,y) = f(z) = \cos(z).$$

- For every $z = x + iy \in \mathbb{C}$ we have

$$\cos(z + 2\pi) = \frac{e^{i(z+2\pi)} + e^{-i(z+2\pi)}}{2} = \frac{e^{iz} + e^{-iz}}{2} = \cos(z), \quad \text{as } e^{2\pi i} = 1,$$
$$-\cos(z + \pi) = -\frac{e^{i(z+\pi)} + e^{-i(z+\pi)}}{2} = -\frac{-e^{iz} - e^{-iz}}{2} = \cos(z), \quad \text{as } e^{\pi i} = -1,$$
$$\sin\left(z + \frac{\pi}{2}\right) = \frac{e^{i(z+\frac{\pi}{2})} - e^{-i(z+\frac{\pi}{2})}}{2i} = \frac{e^{iz}i - e^{-iz}(-i)}{2i} = \frac{e^{iz} + e^{-iz}}{2} = \cos(z), \quad \text{as } e^{\frac{\pi}{2}i} = i.$$

item b

We use $e^{iz} = \cos(z) + i\sin(z)$ and the facts that $\cos(-z) = \cos(z), \sin(-z) = -\sin(z)$, to show that

$$\begin{aligned}\cos(z+w) &= \frac{e^{i(z+w)} + e^{-i(z+w)}}{2} = \frac{e^{iz}e^{iw} + e^{-iz}e^{-iw}}{2} \\ &= \frac{(\cos(z) + i\sin(z))(\cos(w) + i\sin(w)) + (\cos(z) - i\sin(z))(\cos(w) - i\sin(w))}{2} \\ &= \frac{2\cos(z)\cos(w) - 2\sin(z)\sin(w)}{2} = \cos(z)\cos(w) - \sin(z)\sin(w),\end{aligned}$$

that

$$\begin{aligned}\sin(z+w) &= \frac{e^{i(z+w)} - e^{-i(z+w)}}{2i} = \frac{e^{iz}e^{iw} - e^{-iz}e^{-iw}}{2i} \\ &= \frac{(\cos(z) + i\sin(z))(\cos(w) + i\sin(w)) - (\cos(z) - i\sin(z))(\cos(w) - i\sin(w))}{2i} \\ &= \frac{2i\cos(z)\sin(w) + 2i\sin(z)\cos(w)}{2i} = \cos(z)\sin(w) + \sin(z)\cos(w),\end{aligned}$$

that

$$\begin{aligned}\cos^2(z) &= \frac{e^{2iz} + 2 + e^{-2iz}}{4}, \sin^2(z) = \frac{e^{2iz} - 2 + e^{-2iz}}{-4} \implies \cos^2(z) + \sin^2(z) = 1 \\ &\implies \cos(2z) = \cos^2(z) - \sin^2(z) = 1 - 2\sin^2(z) = 2\cos^2(z) - 1\end{aligned}$$

and

$$\sin(2z) = \cos(z)\sin(z) + \sin(z)\cos(z) = 2\sin(z)\cos(z).$$

item c

Already shown in the first part.

Question 5

Item a

We show that f is constant. Assume that a, b are not both 0 (otherwise the equation gives no information). We differentiate the equation by x,y and get:

$$au_x + bv_x = 0,$$
$$au_y + bv_y = 0.$$

Apply C-R $v_x = -u_y,\ v_y = u_x$ and get

$$au_x - bu_y = 0,$$
$$au_y + bu_x = 0.$$

We view this as a set of linear equations where the variables are u_x, u_y and the scalars are a, b. The determinant of this set of equations is $a^2 + b^2$.

- If $a^2 + b^2 \neq 0$, then the only solution is $u_x = u_y = 0$, hence $u(x, y)$ is constant and by C-R, then so is $v(x, y)$ and $f(z)$.
- Otherwise we get $b = \pm ia$, for any $a \neq 0$. Thus $au_x = \pm iu_y$. Both u_x, u_y are real hence the only solution to these equations is still $u_x = u_y = 0$. Again u, v, f are constant.

Item b

i. If $u(x,y) = x^2 - y^2$, then

$$v_y = u_x = 2x \implies v(x,y) = 2xy + F(x),$$
$$-v_x = -2y - F'(x) = u_y = -2y \implies F'(x) = 0 \implies F(x) \equiv C,\ C \in \mathbb{R}$$

thus

$$f(z) = (x^2 - y^2) + i(2xy + C) = (x+iy)^2 + iC = z^2 + iC.$$

iii. If $f(z) = u(x) + iv(y)$, then

$$u'(x) = u_x = v_y = v'(y) \implies u'(x) = v'(y) \equiv C \implies u(x) = Cx + D, v(y) = Cy + E$$
$$u_y = v_x = 0,$$

thus

$$f(z) = (Cx + D) + i(Cy + E) = Cz + (D + iE).$$

iv. If $|f(z)| = e^y$, then $u^2 + v^2 = e^{2y}$, so

$$2uu_x + 2vv_x = 0 \implies uu_x - vu_y = 0,$$
$$2uu_y + 2vv_y = 2e^{2y} \implies uu_y + vu_x = e^{2y}$$

so

$$\begin{pmatrix} u & -v \\ v & u \end{pmatrix}\begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} 0 \\ e^{2y} \end{pmatrix} \implies \begin{pmatrix} u_x \\ u_y \end{pmatrix} = \frac{1}{u^2+v^2}\begin{pmatrix} u & v \\ -v & u \end{pmatrix}\begin{pmatrix} 0 \\ e^{2y} \end{pmatrix} = \begin{pmatrix} v \\ u \end{pmatrix} \implies u_x = v, u_y = u.$$

Now,

$$u_y = u \implies (e^{-y}u)_y = 0 \implies e^{-y}u = F(x) \implies u = e^y F(x),$$

so $v = u_x = e^y F'(x)$. Substitute in the second C-R equation to get

$$u_y = -v_x \implies e^y F(x) = -e^y F''(x) \implies F''(x) + F(x) = 0 \implies F(x) = a\cos(x) + b\sin(x),$$

thus

$$e^{2y} = u^2 + v^2 = e^{2y}F^2(x) + e^{2y}(F')^2(x)$$
$$= e^{2y}(a^2\cos^2(x) + 2ab\cos(x)\sin(x) + b^2\sin^2(x) + a^2\sin^2(x) - 2ab\cos(x)\sin(x) + b^2\cos^2(x))$$
$$= e^{2y}(a^2 + b^2) \implies a^2 + b^2 = 1$$

therefore

$$f(z) = e^y(a\cos(x) + b\sin(x)) + ie^y(-a\sin(x) + b\cos(x)) = e^y(a(\cos(x) - i\sin(x)) + b(\sin(x) + i\cos(x)))$$
$$\implies f(z) = e^y(a+ib)(\cos(x) - i\sin(x)) = (a+ib)e^{-z}$$

so $f(z) = z_0 e^{-z}$ for $|z_0| = 1$.