Selected Answers to HW 3

HW 3

Question 1

Item a

if u is constant, then $u_x, u_y \equiv 0$ in all points in the domain. From the $C-R$ equations, then $v_x = -u_y = 0,\ v_y = u_x = 0$ in all points, thus v is also a constant and so is $f = u + iv$.

Item b

See notes of third review class (or use item c for $p(t_1, t_2) = t_1^2 + t_2^2$).

Item c

[The question didn't say so explicitly, but of course we assume that the polynomial is not a constant, otherwise it is meaningless.] We prove the claim by induction on n, the degree of the polynomial $p(t_1, t_2)$.

- **Base Case:** For $n = 1$, then $p(u, v) = au + bv + c$. See the solution to the last question in HW.
- **Induction Hypothesis:** Assume that for all polynomials $q(t_1, t_2)$ of degree $n > 1$, if $q(u(x, y), v(x, y))$ is constant then u, v are constant.
- **Induction Step:** Let $p(t_1, t_2)$ be a polynomial of degree $n + 1$, we shall prove, assuming the induction hypothesis that if $p(u, v)$ is constant then u, v are constant.

  The function $g(x, y) = p(u(x, y), v(x, y))$ is constant. Hence

  $$\frac{\partial g}{dx} = \frac{\partial g}{dy} = 0$$

  in every point. We use the chain rule:

  $$\frac{\partial g}{dx} = \frac{\partial p}{du}\frac{\partial u}{dx} + \frac{\partial p}{dv}\frac{\partial v}{dx} = 0$$
  $$\frac{\partial g}{dy} = \frac{\partial p}{du}\frac{\partial u}{dy} + \frac{\partial p}{dv}\frac{\partial v}{dy} = 0.$$

  We use C-R to substitute $v_x = -u_y, v_y = u_x$.

  $$\frac{\partial p}{dt_1}u_x - \frac{\partial p}{dt_2}u_y = 0, \tag{0.1}$$
  $$\frac{\partial p}{dt_1}u_y + \frac{\partial p}{dt_2}u_x = 0. \tag{0.2}$$

  For every point $u(x, y), v(x, y)$ you can see this as a linear equation with different coefficients. The determinant of the coefficient matrix of the linear equations is

  $$D(x, y) = \frac{\partial p}{dt_1}^2 + \frac{\partial p}{dt_2}^2.$$

– If ${\frac{\partial p}{dt_1}}^2 + {\frac{\partial p}{dt_2}}^2$ is the zero polynomial, then since p has *real coefficients*, it implies that both

$$\frac{\partial p}{dt_1}\bigg|_{u(x,y),v(x,y)} = \frac{\partial p}{dt_2}\bigg|_{u(x,y),v(x,y)} = 0.$$

At least one of them is a (non-constant) polynomial of degree n, that is equal to zero. By our induction hypothesis, we get that u, v are constant.

– Otherwise $D(x, y)$ is not the zero polynomial, and in particular, this means that the set of pairs (x, y) where it is not equal to 0 is open and dense in the domain. Thus in that set the only solutions to the equations (0.1) is that $u_x, u_y = 0$ in that set. This implies that u, v are constnat in this set. From continuity, we get that they are constant in *all the domain.*

Question 5

Item a

No, take for example

$$f(x + iy) = e^{-x}(cos(y) + isin(y)).$$

Item b

The proof to this question is the same as the proof for the same question in $\mathbb{R}$. Denote $g(z) = e^{-z}f(z)$. Then

$$g'(z) = f'(z)e^{-z} - f(z)e^{-z}.$$

Notice that we are using here the following "differentiation arithmetic":

- $(e^z)' = e^z$
- $(a(z)b(z))' = a'(z)b(z) + a(z)b'(z)$
- $a(b(z)) = a'(b(z))b'(z)$.

From our assumption $f'(z) = f(z)$, thus $g'(z) = 0$, is 0. Hence $g(z)$ is a constant function. From the assumption that $f(0) = 1$, then so does $g(0) = 1$, thus $f(z) = e^{-z}$.

Question 6

Item a

i False. It might not be defined, e.g. $\sqrt{i^2}$.

ii True. The n-th root is an inverse function of z^n, hence if it is defined on z, then by definition $(\sqrt[n]{z})^n = z$.

iii True, $e^{2\pi i} = 1$.

iv False, again this might not be defined, e.g. $z = \sqrt{i}, w = \sqrt{i}$.

v False. e.g. when we represent $-\pi \leq \theta < \pi$ and $\sqrt{re^{i\theta}} = \sqrt{r}e^{i\frac{\theta}{2}}$, and $z = w = e^{i\frac{3\pi}{4}}$.

vi True. If $Re(z), Re(w) > 0$, then $-\pi/2 < arg(z), arg(w) < \pi/2$, and in particular $-\pi < arg(zw) < \pi$, i.e. the n-th root is defined, and the argument doesn't complete a full cycle - namely, the argument of the root will be $\frac{arg(zw)}{n} = \frac{arg(z)+arg(w)}{n}$.

Item b

- Our domain is all $-\pi < \theta < \pi$, and $f(re^{i\theta}) = \sqrt[n]{r}e^{i\theta/n}$. Thus our image is precisely
$$\left\{re^{i\theta} : -\pi/n < \theta < \pi/n\right\}.$$
- If we parametrize $Ray_{a,b} = \left\{re^{i\theta} : r > 0\right\}$ and θ is the argument of the ray, then it is sent to $Ray_{a',b'} = \left\{re^{i\theta/n} : r > 0\right\}$.

Item c

Notice that $(e^{9\pi i/8})^4 \neq \frac{-1+i}{\sqrt{2}}$ thus there was an error in the question.

Item d

The complex inverse function theorem, that is implied by the real inverse function theorem, says that $(f^{-1})'(z) = \frac{1}{f'(f^{-1}(z))}$. Thus $(\sqrt[n]{z})' = \frac{1}{n(\sqrt[n]{z})^{n-1}}$. In particular
$$\frac{zf'(z)}{f(z)} = \frac{z}{n(\sqrt[n]{z})^n} = \frac{1}{n}.$$

Item e
$$\frac{f_1(z)f_2'(z)}{f_1'(z)f_2(z)} = \frac{zf_1(z)f_2'(z)}{f_1'(z)zf_2(z)} = 1,$$
where the last equality is by item d.

Question 7

Item a

From the definition $f : [0, \pi] \to \mathbb{C}$, $f(z) = e^{iz} = cos(z)+isin(z)$. Note that $\sin(z) = 0$ only when $z = 0, \pi$, thus no other real point exists in the image of f. Thus there is no point in $(f(0), f(\pi)) = (-1, 1) \subset \mathbb{C}$ in the image.

Item b

No. take the same function $f(z)$ as above. $\frac{f(\pi)-f(0)}{\pi-0} = \frac{2}{\pi}$. However, $f'(z) = ie^iz$, a function whose norm is always 1, hence $f'(c) \neq \frac{2}{\pi}$ for all $c \in (0, \pi)$.