HOMEWORK SHEET 4: SOLUTIONS

INTRODUCTION TO COMPLEX ANALYSIS FOR ELECTRIC ENGINEERING

1. Question 1:

Let $Log(z)$ be the main branch ("Anaf Rashee") of the logarithm; Recall that $Log : \mathbb{C} \setminus \mathbb{R}_- \to \mathbb{C}$ is given by

$$Log(z) = \ln(r) + i\theta,$$

where $z = re^{i\theta}$ with $-\pi < \theta < \pi$.

a) Let $a, b \in \mathbb{R}$. Write $a + ib = re^{i\theta}$ for $-\pi \leq \theta < \pi$ and $r \geq 0$, then

$$\frac{a+ib}{a-ib} = \frac{re^{i\theta}}{re^{-i\theta}} = e^{2\theta i}.$$

Therefore,

$$Log\left(\frac{a+ib}{a-ib}\right) = Log(e^{2\theta i}) = \begin{cases} 2\theta i & : \text{if } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ (2\theta - 2\pi)i & : \text{if } \frac{\pi}{2} < \theta < \pi \\ (2\theta + 2\pi)i & : \text{if } -\pi < \theta < -\frac{\pi}{2} \end{cases}$$

where $\theta = \arctan\left(\frac{b}{a}\right)$.

b) **Log is $1-1$:** Let $z_1, z_2 \in \mathbb{C} \setminus \mathbb{R}_-$ and suppose that $Log(z_1) = Log(z_2)$. Write both z_1 and z_2 in their polar coordinates $z_1 = r_1 e^{i\theta_1}, z_2 = r_2 e^{i\theta_2}$, thus

$$Log(z_1) = Log(z_2) \Longrightarrow \ln(r_1) + i\theta_1 = \ln(r_2) + i\theta_2 \Longrightarrow \ln(r_1) = \ln(r_2),\ \theta_1 = \theta_2$$

and as ln is $1-1$ in $\mathbb{R}$, we get that $r_1 = r_2, \theta_1 = \theta_2 \Longrightarrow z_1 = z_2$.

- Let $Ray_{a,b} = \{(at, bt) : t \in \mathbb{R}_{>0}\}$, then the image of Log on $Ray_{a.b}$ is given by

$$\begin{aligned} Log(Ray_{a,b}) &= \{Log(at + ibt) : t \in \mathbb{R}_{>0}\} = \{Log(t\sqrt{a^2+b^2}e^{i\theta}) : t \in \mathbb{R}_{>0}\} \\ &= \{\ln(t\sqrt{a^2+b^2}) + i\theta : t \in \mathbb{R}_{>0}\} = \{z : Im(z) = \theta\} \end{aligned}$$

and this is exactly the line in $\mathbb{C}$ where the imaginary part is equal to the constant $-\pi < \theta = \arctan\left(\frac{b}{a}\right) < \pi$.

c)(i) No. Take $z = 2\pi i$, so $Log(e^{2\pi i}) = Log(e^{0i}) = 0 \neq 2\pi i$.

(ii) Yes. Follows from the definition of Log.

(iii) No. As in (i); take $z = 0$, so $Log(e^{2\pi i}) = 0 \neq 2\pi i$.

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(iv) Yes. Let $z = re^{i\theta}$, with $-\pi < \theta < \pi$, then

$$Log\left(\frac{1}{z}\right) = Log\left(\frac{1}{r}e^{-i\theta}\right) = \ln\left(\frac{1}{r}\right) + (-i\theta) = -\ln(r) - i\theta = -Log(z),$$

as $-\pi < \theta < \pi \implies -\pi < -\theta < \pi$.

(v) No. Take $z = w = e^{\frac{3}{4}\pi i}$, then

$$Log(zw) = Log(e^{\frac{3}{2}\pi i}) = Log(e^{-\frac{\pi}{2}i}) = -\frac{\pi}{2}i \neq \frac{3}{2}\pi i = 2Log(e^{\frac{3}{4}\pi i}) = Log(z)+Log(w).$$

(vi) Yes. Suppose $Re(z), Re(w) > 0$, thus we can write $z = r_1e^{i\theta_1}, w = r_2e^{i\theta_2}$ where $-\frac{\pi}{2} < \theta_1, \theta_2 < \frac{\pi}{2}$ and hence

$$\begin{aligned} Log(zw) = Log(r_1r_2e^{i(\theta_1+\theta_2)}) &= \ln(r_1r_2) + i(\theta_1 + \theta_2) \\ &= \ln(r_1) + i\theta_1 + \ln(r_2) + i\theta_2 = Log(z) + Log(w) \end{aligned}$$

where the second equality holds because $-\pi < \theta_1 + \theta_2 < \pi$.

d) No. Suppose there exists $F : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ analytic such that

$$F'(z) = \frac{1}{z}, \forall z \in \mathbb{C} \setminus \{0\}.$$

Consider the function $Log : \mathbb{C} \setminus \mathbb{R}_- \to \mathbb{C}$; we know that for every

$$(Log)'(z) = \frac{1}{z}, \forall z \in \mathbb{C} \setminus \mathbb{R}_-,$$

therefore $(F - Log)'(z) = 0$ for every $z \in \mathbb{C} \setminus \mathbb{R}_-$ and hence

$$\exists C \in \mathbb{C} \text{ such that } F(z) - Log(z) = C, \forall z \in \mathbb{C} \setminus \mathbb{R}_-.$$

From the assumption, the function F is continuous at $z = -1$ and so we must have

$$F(-1) = \lim_{\theta\to\pi^-} F(e^{i\theta}) = \lim_{\theta\to-\pi^+} F(e^{i\theta}),$$

but for every $-\pi < \theta < \pi$, $F(e^{i\theta}) = Log(e^{i\theta}) + C = i\theta + C$, therefore we get

$$F(-1) = \lim_{\theta\to\pi^-}(i\theta + C) = i\pi + C = \lim_{\theta\to-\pi^+}(i\theta + C) = -i\pi + C$$

which is clearly a contradiction.

2. Question 2.

a)(i) Yes. $\sum_{n=0}^{\infty} z_n$ converges $\iff$ the sequence $a_N = \sum_{n=0}^{N} z_n$ converges (by definition) $\iff$ the sequences

$$Re(a_N) = \sum_{n=0}^{N} Re(z_n) \text{ and } Im(a_N) = \sum_{n=0}^{N} Im(z_n)$$

converge (see question 3.a)(i) in Exercise 1) $\iff$ $\sum_{n=0}^{\infty} Re(z_n)$ and $\sum_{n=0}^{\infty} Im(z_n)$ converge (definition of convergence in $\mathbb{R}$).

(ii) Yes. If $\sum_{n=0}^{\infty} z_n$ is absolutely converges, then $\sum_{n=0}^{\infty} |z_n|$ converges. As

$$0 \le |Re(z_n)| \le |z_n| \text{ and } 0 \le |Im(z_n)| \le |z_n|,$$

using the convergence theorems from Calculus in $\mathbb{R}$, we know that $\sum_{n=0}^{\infty} |Re(z_n)|$ and $\sum_{n=0}^{\infty} |Im(z_n)|$ converge $\Longrightarrow \sum_{n=0}^{\infty} Re(z_n)$ and $\sum_{n=0}^{\infty} Im(z_n)$ converge $\Longrightarrow \sum_{n=0}^{\infty} z_n$ converges (from part (i)).

(iii)+(iv) Yes. If R is the radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$, then (as it was seen in class) $\sum_{n=0}^{\infty} a_n z^n$ converges for any $|z| < R$ and diverges for any $|z| > R$. Therefore, if $\sum_{n=0}^{\infty} a_n z^n$ converges, then $|z| \leq R$, and if $\sum_{n=0}^{\infty} a_n z^n$ diverges, then $|z| \geq R$.

b) Let R_a and R_b be the radiuses of convergence of $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$. Then

$$\sqrt[n]{|a_n b_n|} = \sqrt[n]{|a_n|}\,\sqrt[n]{|b_n|} \Longrightarrow \limsup_{n\to\infty} \sqrt[n]{|a_n b_n|} \leq (\limsup_{n\to\infty} \sqrt[n]{|a_n|})(\limsup_{n\to\infty} \sqrt[n]{|b_n|}).$$

- If $0 < R_a, R_b < \infty$, then
$$\frac{1}{R} \leq \frac{1}{R_a}\frac{1}{R_b} \Longrightarrow R \geq R_a R_b.$$
- If $R_a = \infty$ and $0 < R_b$ (or $R_b = \infty$ and $0 < R_a$), then $R = \infty$.
- If $R_a = 0$ or $R_b = 0$, then it could be that $R = 0, \infty$ and $0 < R < \infty$.

c) Let $f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}$, where $s \in \mathbb{R}$. Denote $a_n = \frac{1}{n^s}$, so

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^s = 1,\ \forall s \in \mathbb{R}.$$

Therefore the radius of convergence is $R = 1$.

- Let $s = -2$, then
$$f(z) = \sum_{n=0}^{\infty} n^2 z^n.$$

Notice that the function

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

is analytic in $Ball_1(0)$, which implies that

$$\frac{1}{(1-z)^2} = \left(\frac{1}{1-z}\right)' = \sum_{n=1}^{\infty} n z^{n-1} \Longrightarrow \sum_{n=1}^{\infty} n z^n = \frac{z}{(1-z)^2},\ \forall z \in Ball_1(0)$$

with the same radius of convergence; once again, this implies that

$$\frac{1+z}{(1-z)^3} = \left(\frac{z}{(1-z)^2}\right)' = \sum_{n=1}^{\infty} n^2 z^{n-1} \Longrightarrow \sum_{n=1}^{\infty} n^2 z^n = \frac{z(1+z)}{(1-z)^3}$$

$$\Longrightarrow f(z) = \frac{z(1+z)}{(1-z)^3}.$$

d) Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and plug it in the equation

$$f(z^2) = z + f(z) \Longrightarrow \sum_{n=0}^{\infty} a_n z^{2n} = z + \sum_{n=0}^{\infty} a_n z^n.$$

Compare the coefficients in both sides, we get

$$a_1 = -1,$$
$$a_{2n+1} = 0,\ \forall n \geq 1$$
$$a_n = a_{2n},\ \forall n \geq 1$$

so from these equalities we see that for any $n \geq 1$,

$$a_n = \begin{cases} 0 : n \neq 2^k \\ -1 : n = 2^k \end{cases}$$

so

$$f(z) = a_0 - \sum_{n=0}^{\infty} z^{2^n}$$

with radius of convergence equal to 1 (as $\sqrt[n]{|a_n|} = 1$ for $n = 2^k$ and $\sqrt[n]{|a_n|} = 0$ otherwise).

3. Question 3.

a) Proved in class.

b) We use the fact that $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ for every $z \in \mathbb{C}$, to show that

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} = \frac{1}{2i}\left(\sum_{n=0}^{\infty} \frac{(iz)^n}{n!} - \sum_{n=0}^{\infty} \frac{(-iz)^n}{n!}\right) = \frac{1}{2i}\sum_{n=0}^{\infty} \frac{(iz)^n(1-(-1)^n)}{n!}$$
$$= \frac{2}{2i}\sum_{2\nmid n} \frac{(iz)^n}{n!} = \frac{1}{i}\sum_{k=0}^{\infty} \frac{(iz)^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!}$$

and that

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} = \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{(iz)^n}{n!} + \sum_{n=0}^{\infty} \frac{(-iz)^n}{n!}\right) = \frac{1}{2}\sum_{n=0}^{\infty} \frac{(iz)^n(1+(-1)^n)}{n!}$$
$$= \frac{2}{2}\sum_{2|n} \frac{(iz)^n}{n!} = \sum_{k=0}^{\infty} \frac{(iz)^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!},$$

for every $z \in \mathbb{C}$.

c) We know that

$$(1-z)(1+\ldots+z^N) = 1 - z^{N+1} \Longrightarrow \sum_{n=0}^{N} z^n = \frac{1-z^{N+1}}{1-z},\ \forall N \geq 1$$

and as $\lim_{N\to\infty} z^{N+1} = 0$ for every $|z| < 1$, we get that

$$\lim_{N\to\infty}\left(\sum_{n=0}^{N} z^n\right) = \lim_{N\to\infty} \frac{1-z^{N+1}}{1-z} = \frac{1}{1-z}$$

which means that the sum $\sum_{n=0}^{\infty} z^n$ converges to the function $f(z) = \frac{1}{1-z}$, when $|z| < 1$; this is also the radius of convergence ($R = 1$). Therefore,

$$\frac{1}{z - z_0} = \frac{1}{(z_1 - z_0) + (z - z_1)} = \frac{1}{(z_1 - z_0)(1 + \frac{z - z_1}{z_1 - z_0})} = \frac{1}{z_1 - z_0} \cdot \frac{1}{1 - \frac{z - z_1}{z_0 - z_1}}$$

so for every z such that $\left|\frac{z-z_1}{z_0-z_1}\right| < 1$, we have

$$\frac{1}{z - z_0} = \frac{1}{z_1 - z_0} \cdot \sum_{n=0}^{\infty} \left(\frac{z - z_1}{z_0 - z_1}\right)^n = \sum_{n=0}^{\infty} a_n (z - z_1)^n$$

where

$$a_n = -\left(\frac{1}{z_0 - z_1}\right)^{n+1}.$$

The radius of convergence is given by

$$\frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{|z_0 - z_1|} \Longrightarrow R = |z_0 - z_1|.$$

d) Let $a_n = \frac{(-1)^{n+1}}{n}$ for $n \geq 1$, then it is easily seen that

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{n}{n+1}\right| \longrightarrow 1, \text{as } n \to \infty$$

and so the radius of convergence of $\sum_{n=1}^{\infty} \frac{(-1)^{n+!}}{n} z^n$ is equal to 1. Then we can define the function

$$f(z) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n$$

which is analytic in the ball $Ball_1(0)$ and we know that we have

$$f'(z) = \sum_{n=1}^{\infty} (-1)^{n+1} z^{n-1} = \sum_{k=0}^{\infty} (-z)^k$$

for every $z \in Ball_1(0)$. On the other hand, if $z \in Ball_1(0)$, then $\in Ball_1(0)$ and hence we know that

$$f'(z) = \sum_{k=0}^{\infty} (-z)^k = \frac{1}{1+z} = (Log(z+1))'(z),$$

where $z + 1 \in \mathbb{C} \setminus \mathbb{R}_-$ as $|z| < 1$. This implies that

$$f(z) = Log(z+1) + c$$

for some constant $c \in \mathbb{C}$, however for $z = 0$ we get $f(0) = 0 = Log(1) + c = c$, so

$$f(z) = Log(z+1), \forall z \in Ball_1(0).$$

e) From the previous part we know that

$$Log(1 + z^3) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^{3n}, \forall z \in Ball_1(0)$$

and hence that

$$z^3 Log(1 + z^3) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^{3n+3} = \sum_{k=2}^{\infty} \frac{(-1)^k}{k-1} z^{3k}, \forall z \in Ball_1(0).$$

4. Question 4.

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $a_n \in \mathbb{R}$ and suppose the sum converges in $Ball_r(0)$.
a) Suppose that $f(x) = 0$ for all $x \in (-\epsilon, \epsilon)$ for some $\epsilon > 0$. From the assumption, the function $f(z)$ is analytic in $Ball_r(0)$, so one can write

$$f(z) = u(x, y) + iv(x, y)$$

and we know that u, v satisfy the Cauchy-Riemann equations at every point in $Ball_r(0)$. From the assumption

$$0 = f(x) = u(x, 0) + iv(x, 0),\ \forall x \in (-\epsilon, \epsilon)$$

so $u(x, 0) = v(x, 0) = 0$ for all $x \in (-\epsilon, \epsilon)$ and hence by definition we get that

$$u(0, 0) = v(0, 0) = 0$$

and

$$\frac{\partial^k u}{x^k}(0, 0) = \frac{\partial^k v}{x^k}(0, 0) = 0,\ \forall k \geq 1,$$

but from Cauchy-Riemann we know that

$$f^{(k)}(0) = \frac{\partial^k u}{\partial x^k}(0, 0) + i\frac{\partial^k v}{\partial x^k}(0, 0) = 0,\ \forall k \geq 0.$$

On the other hand, from the analyticity of f, for every $k \geq 0$ we have

$$f^{(k)}(0) = \Big(\sum_{n=k}^{\infty} n(n-1)\cdots(n-(k-1))a_n z^{n-k}\Big)\ |_{z=0} = k!a_k,$$

so $k!a_k = 0$, i.e., $a_k = 0$ for all $k \geq 0$, which means that $f(z) = 0$ for all $z \in Ball_r(0)$.

b) A direct use of the first part (so we skip some details); define the function

$$h(z) = \sum_{k=1}^{n} c_k \frac{\partial^k f(z)}{\partial z^k} + c_0.$$

As $f(z)$ is analytic in $Ball_r(0)$, so is $h(z)$. From the assumption, it follows that $h(x) = 0$ for all $x \in (-\epsilon, \epsilon)$ and hence $h(z) = 0$ for all $z \in Ball_r(0)$.