Selected Answers to HW 5

HW 5

Question 2

First compute $|\left(1+\frac{z}{n}\right)^n| = \left(|1+\frac{z}{n}|\right)^n = \left((1+\frac{x}{n})^2+(\frac{y}{n})^2\right)^{n/2}$.

Thus: $\lim_{n\to\infty}|\left(1+\frac{z}{n}\right)^n| = \lim_{n\to\infty}\left(1+\frac{2x}{n}+\frac{x^2+y^2}{n^2}\right)^{n/2} = e^x$.

Now the argument: $Arg(1+\frac{z}{n}) = Arctan\frac{y/n}{1+x/n}$, thus by De Moivre, $Arg\left(1+\frac{z}{n}\right)^n = n\cdot Arctan\frac{y/n}{1+x/n}$.

Thus $\lim_{n\to\infty} Arg\left(1+\frac{z}{n}\right)^n = \lim_{n\to\infty} n\cdot Arctan\frac{y/n}{1+x/n} = y$.

(Note: arctan is defined up to π, but for $n \gg 1$: $1+\frac{z}{n}\sim 1$, thus we choose the principal branch, $|Arctan| < \pi/2$)

Question 3

Item a

Our goal is to construct $\delta(t)$ so that $\gamma(t) = e^{\delta(t)}$. Intuitively we would like to say that $\delta(t) =$ "$log\gamma(t)$". However, we have no guarantee that any branch of log is defined on all γ. Instead we define $log\gamma(t)$ locally, and stitch the pieces. Details follow:

Note that since $0\notin\gamma$, and γ is compact (closed and bounded), the continuous function $t\mapsto|\gamma(t)|$ has a minimum. Denote this minimum ε. By the hint, there is a partition of the interval $0 = t_1 < ... < t_n = 1$, so that $\gamma([t_i, t_{i+1}])$ is in a some ball of radius $\varepsilon/2$ around $\gamma(t_i)$. Since this ball doesn't contain 0, it also doesn't contain any path that circles 0 (the ball is convex, hence if it contained a path that circles 0 it must have contained 0 itself). Hence locally for every i, there exists some branch of log, $f_i : Ball_{\varepsilon/2}(t_i)\to\mathbb{C}$. Now we define $\delta(t)$ inductively on each interval:

1. On $[t_0, t_1]$ we define $\delta(t) = f_0(\gamma(t))$.
2. Given that we defined $\delta(t)$ on $[t_0, t_j]$, we need to define it on $[t_j, t_{j+1}]$. The branches f_j, f_{j+1} are both defined on the point $\gamma(t_j)$. We saw in class that if they are both defined on some shared point, then $f_{j+1}(\gamma(t_j)) - f_j(\gamma(t_j)) = 2\pi i k_j$ for some integer k_j. We define $\delta(t) = f_{j+1}(\gamma(t)) - 2\pi i k_j$ (and note that by deducting $2\pi i k_j$ we still get a branch of log).

By construction our function is continuous, and since on every point $\delta(t) = f_i(\gamma(t)) + 2\pi i k_j$ for some k_j and branch f_j of log, then $\gamma(t) = e^{\delta(t)}$.

Item b

Since $\gamma(0) = \gamma(1)$ then $e^{\delta(0)} = e^{\delta(1)}$. By what we saw in class, this means that $\delta(1) - \delta(0)$ is $2\pi i k$ for some integer $k\in\mathbb{Z}$.

Equality doesn't necessarily hold (i.e. the path is not always closed). Even in the case that $\gamma(t) = e^{2\pi i t}$, then we can define $\delta(t) = 2\pi i t$ and get $\delta(0) = 0$ and $\delta(1) = 2\pi$.

Item c

Since $e^{\delta(t)} = e^{\tilde{\delta}(t)}$, then the difference $\tilde{\delta}(t) - \delta(t) = 2\pi i k$ for some integer k. In addition $\tilde{\delta}(t) - \delta(t)$ is continuous, and since this is a continuous function with integral values (up to scaling by $2\pi i$), then it must be a constant.

Note that this is the most we can say since if $\delta(t) + 2\pi i k$ is also a path that has $e^{\delta(t)+2\pi i k} = \gamma(t)$ for every integer k.

Question 4

Item a

We saw in class that

$$\left|\int_\gamma f(z)dz\right| \leq \max_{z\in\gamma}\{|f(z)|\}length(\gamma).$$

Indeed in this case $length(\gamma_{R,\Phi_1,\Phi_2}) \leq 2\pi R$. Thus if $\max_{|z|=R}\{|f(z)|\} < \frac{C}{R^{1+\varepsilon}}$ then

$$\left|\int_{\gamma_{R,\Phi_1,\Phi_2}} f(z)dz\right| \leq \frac{2\pi RC}{R^{1+\varepsilon}} \xrightarrow[R\to\infty]{} 0.$$

Item b

No. For example take $p(z) = 1$ and $q(z) = z$. The integral of

$$\int_{\gamma_{R,\Phi_1,\Phi_2}} \frac{dz}{z} = \int_{\Phi_1}^{\Phi_2} \frac{Rie^{it}dt}{Re^{it}} = \Phi_2 - \Phi_1.$$

In particular it doesn't go to 0.

Item c

To prove that $\lim_{R\to\infty} \int_{R,\Phi_1,\Phi_2} e^{iaz} f(z)dz = 0$, it is enough to prove that $\int_{R,\Phi_1,\Phi_2} |e^{iaz}||dz|$ is bounded since

$$\left|\int_{R,\Phi_1,\Phi_2} e^{iaz} f(z)dz\right| \leq \int_{R,\Phi_1,\Phi_2} |e^{iaz}||f(z)||dz| \leq \max_{R,\Phi_1,\Phi_2} |f(z)| \int_{R,\Phi_1,\Phi_2} |e^{iaz}||dz|.$$

Consider the parametrization $\gamma : [\Phi_1, \Phi_2] \to \mathbb{C},\ \gamma(t) = Re^{it}$.

$$\int_{R,\Phi_1,\Phi_2} |e^{iaz}||dz| = \int_{\Phi_1}^{\Phi_2} |e^{iaRe^{it}}||Rie^{it}|dt = R\int_{\Phi_1}^{\Phi_2} |e^{iaRe^{it}}|dt.$$

The norm of $e^{iaRe^{it}}$ is $e^{Re(iaRe^{it})} = e^{-aRsin(t)}$.

- **First solution:** In the range $[0, \pi]$ the sine function is *concave* (i.e. -*sin(x)* is convex), thus the line between $(0, \sin(0))$ and $\pi, \sin(\pi)$ in under the graph of $sin(x)$ (prove this...). Thus $sin(t) \geq \frac{2t}{\pi}$. Thus $e^{-aRsin(t)} \leq e^{-\frac{2aRt}{\pi}}$. And

$$R\int_{\Phi_1}^{\Phi_2} |e^{iaRe^{it}}|dt \leq R\int_{\Phi_1}^{\Phi_2} e^{-\frac{2aRt}{\pi}}dt = \frac{\pi}{2a}e^{-\frac{2aR\Phi_2}{\pi}} - e^{-\frac{2aR\Phi_1}{\pi}} \leq \frac{\pi}{2a}.$$

- **Second solution:**

  1. If $\Phi_1 > 0$ then $\sin(t) \geq \sin(\Phi_1) > 0$ thus $e^{-aRsin(t)} \leq e^{-asin(\Phi_1)R}$ in all the domain, and thus

$$R\int_{\Phi_1}^{\Phi_2} |e^{iaRe^{it}}|dt \leq Re^{-aRsin(\Phi 1)}(\Phi_2 - \Phi_1) \xrightarrow[R\to\infty]{} 0$$

2. Otherwise $\Phi_1 = 0$. We now that $\lim_{x\to 0} \frac{sin(x)}{x} = 1$, thus there is a small (real) half-open interval that contains 0 where $sin(t) \geq \frac{t}{2}$. Denote this interval $[0, \ell)$. Thus we can write

$$R\int_0^{\Phi_2} |e^{iaRe^{it}}|dt = R\int_0^{\ell} |e^{iaRe^{it}}|dt + R\int_{\ell}^{\Phi_2} |e^{iaRe^{it}}|dt.$$

The part $R\int_{\ell}^{\Phi_2} |e^{iaRe^{it}}|dt$ goes to 0 as in the first case. The part

$$R\int_0^{\ell} |e^{iaRe^{it}}|dt \leq R\int_0^{\ell} e^{-aR\frac{t}{2}}dt = R\frac{1}{2aR}(1 - e^{-aR\frac{\ell}{2}}),$$

which is at most $\frac{1}{2a}$.

Question 5

Item b

Recall that Green's Theorem from calculus gives us that

$$\int_{\partial U} P(x,y)dx + Q(x,y)dy = \iint_U \left(\frac{\partial Q}{dx} - \frac{\partial P}{dy}\right) dxdy.$$

In particular, for $(P,Q) = (-y,x)$ then

$$\int_{\partial U} P(x,y)dx + Q(x,y)dy = 2\iint_U 1dxdy = 2area(U).$$

Consider the integral $\int_{\partial U} \bar{z}dz$. Note that for any $f(z) = u(z) + iv(z)$, the complex integral of $f\bar{(z)}$ is:

$$\int f\bar{(z)}dz = \int (udx + vdy + i\int -vdx + udy.$$

In particular for $f(z) = z = Re(z) + iIm(z)$,

$$\int_{\partial U} \bar{z} = \int xdx + ydy + i\int ydx - xdy = 0 + 2iarea(U).$$

Question 6

Let F_1, F_2 be primitive functions of a function f. Then the derivative of the function $g = F_1(z) - F_2(z)$ is $g'(z) = f(z) - f(z) = 0$. In particular this means that the partial derivatives of g as a function from $\mathbb{R}^2$ to itself are zero for all $z \in U$. We saw in calculus that this means g is constant.