Selected Answers to HW 6

Question 1

Item b

- $f(z) = \frac{cos(z)}{z}$ this function has a point where it is not contiuous (i.e. $z = 0$). Thus we cannot expect it to be analytic in all $\mathbb{C}$.

  1. In $\mathbb{C} \setminus \{0\}$ and $\{|Im(z)| < 1\}$ we have $\gamma = \left\{z : |z| = \frac{1}{2}\right\}$ that circles 0 and by Cauchy's integral formula
  $$\int_\gamma \frac{cos(z)}{z} dz = 2\pi i cos(0) = 2\pi i \neq 0.$$
  Thus by a theorem we saw in class, this function cannot have a primitive function in these domains.
  2. On all other domains ($\{|Im(z)| > 11\}$, $Ball_{\frac{1}{2}}(e^{\frac{2\pi i}{3}})$ and $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$) are simply connected, and $\frac{cos(z)}{z}$ is analytic in them, hence by a theorem we studied in class there exists a primitive function. A formula to a primitive function is given by
  $$F(z) = \int_{\gamma_{w_0,z}} f(u) du$$
  for some fixed w_0 in the domain, and any choice of path γ from w_0 to z contained in the domain.

- $f(z) = \frac{sin(z)}{z}$: As $sin(0) = 0$, the point 0 has order 1 for $sin(z)$ and thus $\frac{sin(z)}{z}$ is analytic in all $\mathbb{C}$. As this domain is simply connected, it has a primitive function as we saw in class.

- $f(z) = \frac{e^z - 1}{z}$ is the same as above.

Item d

Let $f : U \to \mathbb{C}$ be as in the question. We claim that there is an open set W so that $\gamma \subset W \subset U$ where $|f(z) - 1| < 1$.

Indeed, suppose there is not. This means that for any open set containing γ, there is a point where $|f(z) - 1| \geq 1$. In particular, as γ is compact inside U, the open set
$$W_n = U \bigcup_{x \in \gamma} Ball_{\frac{1}{n}}(x),$$
contains a point $w_n \in W_n$ so that $|f(w_n) - 1| \geq 1$. From Wierstrass's theorem, w_n is bounded hence it has a subsequence that converges to w_0 and from the way we constructed w_n then $w_0 \in \gamma$. From contiuity this means that $|f(w_0) - 1| \geq 1$ which is a contradiction.

Thus without loss of generality we may assume that $|f(z) - 1| < 1$ in all the domain of f (by looking at its restriction to W.

Now consider the function $g(z) = Log(f(z))$. Since $f(z) \notin (-\infty, 0]$ (as for all $w \in (-\infty, 0]$ we know that $|w - 1| \geq 1$), $g(z)$ is well defined in the domain of f.

$g(z)$ is analytic and its derivative is $g(z) = \frac{f'(z)}{f(z)}$. Thus from the fundamental theorem of calculus for the complex numbers

$$\oint_\gamma g'(z)dz = \oint_\gamma \frac{f'(z)}{f(z)}dz = 0.$$

Question 2

Item c

1. Notice that
$$\frac{1}{z^2+1} = \frac{1}{2}\left(\frac{1}{iz+1} + \frac{1}{-iz+1}\right),$$
thus we define the function $F(z) = \frac{1}{2i}\left(Log(iz+1) - Log(-iz+1)\right)$. Note that this function is defined in the domain $\mathcal{D}$, $F(0) = 0$ and its derivative is $\frac{1}{z^2+1}$.

   Note: Since our domain D is simply connected and f is analytic, we can deduce the primitive function without calculating its formula explicitly. The reason we calculated it is for the next item.

2. As we know that $tan(0) = F(0) = 0$, it is enough to show that $F(tan(z)) - z$ is constant, which is equivalent of showing that its derivative is 0, or that
$$F'(tan(z)) = 1.$$
Indeed,
$$F'(tan(z)) = \frac{1}{1+tan^2(z)}tan'(z) = \frac{1}{cos^2(z)(1+tan^2(z))} = \frac{1}{cos^2(z)+sin^2(z)} = 1.$$

Question 3

Item a

We will prove for $n \geq 1$ that $\eta(\gamma_n, z_0) = n$. For negative n the proof is similar (as it is the same as for positive n except that the path is in the opposite direction). Indeed, notice that the path $\gamma_n = z_0 + e^{2\pi i t n}$ is a concatenation of the path $z_0 + e^{2\pi i t}$ to itself n times. Thus by item f below

$$\eta(\gamma_n, z_0) = n\eta(\gamma_1, z_0).$$

By direct calculation we did in class $\eta(\gamma_1, z_0) = \int_{|z-z_0|=R} \frac{dz}{z-z_0} = 1$ and we're done.

Item e

recall that in exercise 5 we showed that we can write any path $\gamma(t)$ as $\gamma(t) = z_0 + e^\delta(t)$. Thus

$$\eta(\gamma, z_0) = \frac{1}{2\pi i}\int_\gamma (t)\frac{dz}{z-z_0} = \frac{1}{2\pi i}\int_0^{2\pi} \frac{\delta'(t)e^{\delta(t)}dt}{e^{\delta(t)}} = \frac{1}{2\pi i}\int_0^{2\pi} \delta'(t)dt = \frac{1}{2\pi i}(\delta(2\pi) - \delta(0)).$$

We also saw in the last exercise that $\delta(2\pi) - \delta(0) = 2\pi i k$ for some integer k. Thus the value is always an integer.

Item f

By linearity of integration,

$$\eta(\gamma_1 * \gamma_2, z_0) = \int_{\gamma_1*\gamma_2} \frac{1}{z-z_0}dz = \int_{\rho_1} \frac{1}{z-z_0}dz + \int_{\rho_2} \frac{1}{z-z_0}dz$$

Where $\rho_1 : [0, \frac{1}{2}] \to \mathbb{C}$ is $\rho_1(t) = \gamma_1(2t)$ and $\rho_2 : [\frac{1}{2}, 1] \to \mathbb{C}$ is $\rho_2(t) = \gamma_2(2t - 1)$. Hence

$$\int_{\rho_1} \frac{1}{z - z_0} dz = \int_0^{\frac{1}{2}} \frac{\rho_1'(2t)}{\rho_1(t) - z_0} dt = \int_0^{\frac{1}{2}} \frac{2\gamma_1'(2t)}{\gamma_1(2t) - z_0} dt.$$

By a change of variables $t \mapsto 2u$ we get

$$= \int_0^1 \frac{\gamma_1'(u)}{\gamma_1(u) - z_0} du = \int_{\gamma_1} \frac{1}{z - z_0} dz = \eta(\gamma_1, z_0).$$

Similarly to the second part. Thus

$$\eta(\gamma_1 * \gamma_2, z_0) = \eta(\gamma_1, z_0) + \eta(\gamma_2, z_0).$$