HOMEWORK SHEETS 7: SOLUTIONS

INTRODUCTION TO COMPLEX ANALYSIS FOR ELECTRIC ENGINEERING

Question 1.

a) See question 3 in Tirgul 9 for a solution; Notice that if f is analytic then $|f|^2$ is $\mathbb{R}^2-$differentiable, however $e^{2|z|}$ is not and therefore such an f does not exist.

b) Let $z_n = r_n e^{i\theta_n}$, with $\theta_n \in [-\alpha, \alpha]$ and $\alpha \in (0, \pi/2)$. If $\sum_{n=0}^{\infty} |z_n|$ converges, then $\sum_{n=0}^{\infty} z_n$ converges (see question $2.a(ii)$ in exercise 4).

On the other hand, suppose that $\sum_{n=0}^{\infty} z_n$ converges, thus $\sum_{n=0}^{\infty} r_n \cos(\theta_n)$ converges. However, for every $n \geq 0$ we have $r_n \cos(\theta_n) \geq r_n \cos(\alpha) > 0$, since $\alpha < \pi/2$, so we get that $\sum_{n=0}^{\infty} r_n \cos(\alpha)$ converges, i.e., that $\sum_{n=0}^{\infty} r_n = \sum_{n=0}^{\infty} |z_n|$ converges.

c) Yes. Similarly, if $\sum_{n=0}^{\infty} z_n$ converges, then $\sum_{n=0}^{\infty} r_n \sin(\theta_n)$ converges, but $r_n \sin(\theta_n) \geq r_n \sin(\alpha) > 0$ implies that $\sum_{n=0}^{\infty} r_n$ converges.

d) As $0 \leq r < 1$, we have

$$\sum_{n=0}^{\infty} r^n \cos(n\theta) + i \sum_{n=0}^{\infty} r^n \sin(n\theta) = \sum_{n=0}^{\infty} r^n e^{in\theta} = \frac{1}{1 - re^{i\theta}} = \frac{1 - re^{-i\theta}}{(1 - re^{i\theta})(1 - re^{-i\theta})}$$
$$= \frac{1 - r\cos(\theta) - ir\sin(\theta)}{1 - 2r\cos(\theta) + r^2}$$

and therefore

$$\sum_{n=0}^{\infty} r^n \cos(n\theta) = Re\left(\frac{1 - r\cos(\theta) + ir\sin(\theta)}{1 - 2r\cos(\theta) + r^2}\right) = \frac{1 - r\cos(\theta)}{1 - 2r\cos(\theta) + r^2}$$

and

$$\sum_{n=0}^{\infty} r^n \sin(n\theta) = Im\left(\frac{1 - r\cos(\theta) + ir\sin(\theta)}{1 - 2r\cos(\theta) + r^2}\right) = \frac{r\sin(\theta)}{1 - 2r\cos(\theta) + r^2}.$$

Question 2.

a) We proved in class that the power series converges uniformly on any compact set in $Ball_R(0)$. In particular it converges uniformly on γ (those who don't remember this can recall that any compact set $K \subset Ball_\varepsilon(0)$ has that $k = \max_{z \in K} |z| < R$, thus $\sum_{n=0}^{\infty} |a_n z^n| \leq C \sum_{n=0}^{\infty} \frac{k^n}{R^n}$ for some constant C).

If $f_n \to f$ converges uniformly, then by the integral triangle inequality

$$\left| \int_\gamma (f_n(z) - f(z)) dz \right| \leq \max_{z \in \gamma} |f_n(z) - f(z)| \cdot Length(\gamma).$$

Thus the integrals converge to each other as well.

Hence the integral of the sequence $\left\{\sum_{n=0}^{N} a_n z^n\right\}_N$ converges to the integral of $f(z)$ on γ, and this integral also converges to $\sum_{n=0}^{\infty} a_n \int_\gamma z^n dz$. In particular, this series of complex numbers converges.

b) This exercise shows an interesting relation between Fourier analysis and complex numbers.

Define the following function $g(\theta) = f(re^{i\theta})$. This is a continuous periodic function $g : [-\pi, \pi] \to \mathbb{C}$. Notice that by power series of f, $g(\theta) = f(re^{i\theta}) = \sum_{n=0}^{\infty} a_n r^n e^{in\theta}$. Thus by the uniqueness theorem of Fourier analysis, the Fourier coefficients of $g(z)$ are

$$\hat{g}(n) = \begin{cases} a_n r^n & n \geq 0 \\ 0 & n < 0 \end{cases}.$$

In particular, by Parseval's identity,

$$\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \int_0^{2\pi} |g(\theta)|^2 d\theta = 2\pi \sum_{n=0}^{\infty} |a_n|^2 r^{2n}.$$

Remark 0.1. *One can ask, how come we have no negative Fourier coefficients - we will see the negative coefficients when we learn about Laurent series.*

c) Note that if $h, g : D \to \mathbb{C}$ are defined by a power series in the domain $g(z) = \sum_{n=0}^{\infty} a_n z^n$ and $h(z) = \sum_{n=0}^{\infty} b_n z^n$ then

$$g(z)h(z) = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} a_j b_{n-j} \right) z^n, \tag{0.1}$$

which also converges in all the domain. To show this just note that the n'th derivative of $g(z)h(z)$ at 0 is

$$\sum_{j=0}^{n} \frac{n!}{k!(n-j)!} f^{(j)}(0) g^{(n-j)}(0) = n! \sum_{j=0}^{n} a_j b_{n-j}$$

by Liebnitz's formula. Moreover, we saw that if $g(z)h(z)$ is defined in a ball, then its power series converges in that ball. Thus if we show that $g(z) = \frac{1}{1-z}$ and $h(z) = \int_{\gamma_z} f(z)dz$ are both analytic in $Ball_1(0)$, then their product $F(z) = g(z)h(z)$ is analytic in that ball. If we show that

$$g(z) = \sum_{n=1}^{\infty} \frac{a_{n-1}}{n} z^n$$

and use the fact that

$$h(z) = \sum_{n=0}^{\infty} z^n$$

then by formula (0.1) we get that

$$F(z) = h(z)g(z) = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \frac{a_j}{j+1} \right) z^{n+1}.$$

Indeed we saw in class that $h(z)$ is analytic, and that $g(z)$ is also analytic (and it is in fact a primitive function to $f(z)$). By $g(z)$ being a primitive function to $f(z)$ so that $g(0) = 0$ we can deduce that

$$g(z) = \sum_{n=1}^{\infty} \frac{a_{n-1}}{n} z^n$$

and we are done.

d) Let K be any compact set that doesn't contain any natural number. Denote by $M = \max |z| : z \in K$ (K is bounded hence M exists). Since K is bounded, there exists some natural number N so that $N > M$. Denote by $m = \min |z - n| : z \in K, n < N$ and note that $m > 0$ (since this is a minimum between a finite number of $n \in \mathbb{N}$ and since K doesn't contain any natural number)[1]. Notice that

$$\sum_{n=1}^{\infty} \left| \frac{1}{z-n} + \frac{1}{n} \right| = \sum_{n=1}^{\infty} \frac{|z|}{|n(z-n)|} \leq |M| \sum_{n=1}^{N} \frac{1}{nm} + \sum_{N+1}^{\infty} \frac{1}{n(M-n)}$$

thus by Wierstrass's M-convergence test, this series of functions converges absolutely and uniformly in K. Next we calculate

$$f(z+1) - f(z) = \sum_{n=1}^{\infty} \frac{1}{z+1-n} - \frac{1}{z-n} = \sum_{n=1}^{\infty} \frac{1}{z-(n-1)} - \frac{1}{z-n}.$$

This is a telescopic sequence that is equal to $\frac{1}{z+1}$. Finally we calculate the derivative of $f(z)$:

(1) **First solution:** Consider the series of derivatives $g(z) = \sum_{n=1}^{\infty} \frac{-1}{(z-n)^2}$. This series also converges absolutely and uniformly in K (by the same reasoning as above). By a theorem in calculus that we learned, if $f(z) = \sum_{n=1}^{\infty} f_n(z)$ and $g(z) = \sum_{n=1}^{\infty} f_n'(z)$ converge uniformly, then $f'(z) = g(z)$ (and in particular has a derivative). We proved this theorem in the context of real numbers but the proof in the context of complex numbers is the same when we use the complex integral (check at home please).

(2) **Second solution:** We prove that $g(z) = \sum_{n=1}^{\infty} \frac{-1}{(z-n)^2}$ is the derivative directly. In other words we need to show for any z that

$$\lim_{h \to 0} \frac{f(z+h) - f(z) - hg(z)}{h} = 0$$

Indeed

$$\frac{f(z+h) - f(z) - hg(z)}{h} = \frac{1}{h} \sum_{n=1}^{\infty} \frac{-h}{(z-n)^2 + h(z-n)} + \frac{h}{(z-n)^2} = -h \sum_{n=1}^{\infty} \frac{1}{(z-n)^2(z-n+h)}.$$

As the series $\sum_{n=1}^{\infty} \frac{1}{(z-n)^2(z-n+h)}$ converges, then

$$\lim_{h \to 0} \frac{f(z+h) - f(z) - hg(z)}{h} = 0.$$

[1] In fact, one can show that the distance between a compact set and a closed set is always positive

Question 3.

a) On the one hand, if $f(z)$ is analytic in a ball of radius R then for every $r < R$ by the Cauchy formula and the integral triangle inequality

$$|a_n| = \left| \int_{|z|=r} \frac{f(z)}{z^{n+1}} dz \right| \leq \frac{C}{r^n},$$

For some fixed constant C. In particular the radius of convergence is

$$\frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}} \geq r,$$

so it is at least R. On the other hand, the radius of convergence cannot be more than R: Assume that it was. Then it means that the function converges *uniformly* in the closed ball of radius R. Thus the function is defined and continuous in the closed ball of radius R. Thus

$$\sum_{n=0}^{\infty} a_n z_0^n = \lim_{n\to\infty} f(z_n)$$

which doesn't exist. A contradiction.

b) The power series of $\sin(z)$ around 0 is

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}.$$

The power series of $\frac{1}{z-\pi}$ around 0 can be derived from the series of $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$:

$$\frac{1}{z-\pi} = \frac{-1}{\pi} \frac{1}{1-(z/\pi)} = \sum_{n=0}^{\infty} \frac{-1}{\pi^{n+1}} z^n.$$

The convolution formula for a power series is

$$\left(\sum_{n=0}^{\infty} a_n z^n\right)\left(\sum_{n=0}^{\infty} b_n z^n\right) = \sum_{n=0}^{\infty} \sum_{j=0}^{n} a_j b_{n-j}.$$

By using the convolution formula we get that

$$\frac{\sin(z)}{z-\pi} = \left(\frac{(-1)^n}{(2n+1)!} z^{2n+1}\right)\left(\sum_{n=0}^{\infty} \frac{-1}{\pi^{n+1}} z^n\right) = \sum_{n=0}^{\infty} c_n z_n,$$

where

$$c_n = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(-1)^j}{(2j+1)!} \frac{1}{\pi^{n+1-(2j+1)}}.$$

(Note that because the coefficients of $\sin(z)$ are zero when n is odd, the sum above doesn't really go all the way up to n.) It is hard to calculate the convergence radius of this series directly, but we saw in class that if a function is analytic in a ball, then the series converges in the whole ball. The function $\frac{sin(z)}{z-\pi}$ is analytic in all $\mathbb{C}$, thus the function's series converges in all $\mathbb{C}$.

c) Consider the function $tan(i\sin(z))$. The tangent is defined whenever $z \neq \frac{\pi}{2}+k\pi$. Hence $tan(isin(z))$ is defined whenever

$$isin(z) = \frac{e^{iz}-e^{-iz}}{2} \neq \frac{\pi}{2} + k\pi$$

or

$$e^{iz} - e^{-iz} \neq \pi + 2\pi k.$$

Let z_0 be so that $e^{iz_0} - e^{-iz_0} = \pi + 2\pi k$. Then $\lim_{z\to z_0} tan(isin(z)) = \infty$. Thus by item a of this question the convergence radius must be $|z_0|$ for the smallest z_0 as above.

QUESTION 4.

a) From Cauchy's theorem

$$f(z_0) = \frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f(z)dz}{z-z_0}$$

and using the parameterization $\gamma(t) = z_0 + re^{it}, t \in [0, 2\pi)$, we have

$$f(z_0) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0+re^{it})\gamma'(t)dt}{re^{it}} = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{it})dt,$$

while we know that $f(z_0 + re^{it}) \in \mathbb{R}$ for all $0 \le t < 2\pi$, therefore $f(z_0) \in \mathbb{R}$.

b) Let $g(z) := f(z)f(-z)$, as $f \in \mathcal{O}(\overline{Ball_r(0)})$ we also have $g \in \mathcal{O}(\overline{Ball_r(0)})$ and notice that for every $|z| = r$ the points z and $-z$ are in two different half planes: one in the upper one and one in the lower one, so in any case $|g(z)| \le ab.$. Thus, apply Cauchy's theorem

$$f(0)^2 = g(0) = \frac{1}{2\pi i}\int_{|z|=r} \frac{g(z)dz}{z}$$

and hence

$$|f(0)|^2 = \frac{1}{2\pi}\left|\int_{|z|=r} \frac{g(z)dz}{z}\right| \le \frac{1}{2\pi}\int_{|z|=r} \frac{|g(z)|}{r}|dz| \le \frac{ab}{2\pi r}\int_{|z|=r} |dz| = ab,$$

so $|f(0)| \le \sqrt{ab}$.

QUESTION 5.

a) To show that f is a polynomial of degree $\le k$, we show that all of its Taylor coefficients are equal to 0 for $n > k$. As $f \in \mathcal{O}(\mathbb{C})$, we can write its Taylor series expansion around 0, that is

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

where

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i}\int_{|z|=R} \frac{f(z)}{z^{n+1}}dz, \forall R > 0.$$

Therefore, we get that

$$|a_n| \le \frac{1}{2\pi}\int_{|z|=R} \frac{|f(z)|}{|z|^{n+1}}|dz| \le \frac{1}{2\pi R^{n+1}}\int_{|z|=R} (a + b|z|^k)|dz| = \frac{a+bR^k}{R^n}.$$

If $n > k$, then letting $R \to \infty$, we obtain $|a_n| = 0$. Thus

$$f(z) = \sum_{n=0}^{k} a_n z^n.$$

b) As $\tau \in \mathbb{C} \setminus \mathbb{R}$, we have $\tau = \tau_1 + i\tau_2$, with $\tau_1, \tau_2 \in \mathbb{R}$ and $\tau_2 \neq 0$. Then for every $z \in \mathbb{C}$, there exists $n \in \mathbb{Z}$ such that $0 \leq Im(z - n\tau) < |\tau_2|$ and there exists $m \in \mathbb{Z}$ such that $0 \leq Re(z - n\tau - m) < 1$. Thus, we get that

$$0 \leq Re(z - n\tau - m) < 1,\ 0 \leq Im(z - n\tau - m) < |\tau_2|,$$

i.e., that $z - n\tau - m \in K := \{a + ib : a \in [0, 1], b \in [0, |\tau_2|].\}$

As $f \in \mathcal{O}(\mathbb{C})$, we know that f is continuous in the compact set K and so f is bounded in K. On the other hand, for every $z \in \mathbb{C}$ there exist $n, m \in \mathbb{Z}$ such that $z - n\tau - m \in K$ and together with $f(z) = f(z - n\tau) = f(z - n\tau - m)$ we get that f is bounded (by the same bound) in $\mathbb{C}$. Finally, f is entire and bounded, therefore f is constant, from Liouville's theorem.

c) Suppose there exists $z_0 \in \mathbb{C}$ and $\varepsilon > 0$ such that $|f(z) - z_0| > \varepsilon$ for all $z \in \mathbb{C}$. Then the function $g(z) := f(z) - z_0$ also satisfies $g \in \mathcal{O}(\mathbb{C})$ and $g(z) \neq 0$ for all $z \in \mathbb{C}$, therefore the function $h(z) := 1/g(z)$ is in $\mathcal{O}(\mathbb{C})$ and satisfies $|h(z)| < 1/\varepsilon$ for all $z \in \mathbb{C}$. Then Liouville's theorem implies that $h(z)$ is constant which implies that $g(z)$ is constant and so is $f(z)$; a contradiction.

Therefore, for every $z_0 \in \mathbb{C}$ and $\varepsilon > 0$, there exists $z \in \mathbb{C}$ such that $|f(z) - z_0| \leq \varepsilon$, i.e., the image of f intersect every $\overline{Ball_\varepsilon(z_0)}$ in $\mathbb{C}$ and hence $\overline{f(\mathbb{C})} = \mathbb{C}$.

Question 6.

Recall that if

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

then $ord_{z_0}(f) := \inf\{n \geq 0 : f^{(n)}(z_0) \neq 0\} = \inf\{n \geq 0 : a_n \neq 0\}$, as $a_n = \frac{f^{(n)}(z_0)}{n!}$. Therefore it is enough to know what is the first non-zero coefficient in the Taylor series expansion of f around z_0, to know of what order z_0 is zero of f.

a) We know the Taylor series expansions of $Log(1 + z)$ and $\sin(z)$ around 0, so

$$Log(1 + z^5) = z^5 - \frac{z^{10}}{2} + \frac{z^{15}}{3} - \dots$$

and

$$\sin(z^7) = z^7 - \frac{z^{21}}{6} + \dots,$$

which implies that

$$f(z) = z^2 \left(z^5 - \frac{z^{10}}{2} + \frac{z^{15}}{3} - \dots \right) - \left(z^7 - \frac{z^{21}}{6} + \dots \right) = -\frac{z^{12}}{2} + \dots$$

That means that $f(z) = z^{12} g(z)$ where g is analytic and $g(0) \neq 0$, i.e., $ord_0(f) = 12$.

b) As $f \in \mathcal{O}(Ball_r(0))$, one can write f in its Taylor series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,\ \forall z \in Ball_r(0)$$

where $a_n = \frac{1}{n!} f^{(n)}(0)$. However, $ord_0(f) = \infty$ means that $f^{(n)}(0) = 0$ for all $n \geq 0$, therefore we get that $f(z) = 0$ for all $z \in Ball_r(0)$.

c) If $ord_{z_0}(f) = p$, then $a_p \neq 0$ and $a_n = 0$ for any $n < p$; thus

$$f(z) = \sum_{n=p}^{\infty} a_n (z-z_0)^n = \sum_{k=0}^{\infty} a_{p+k}(z-z_0)^{p+k} = (z-z_0)^p \sum_{k=0}^{\infty} a_{p+k}(z-z_0)^k = (z-z_0)^p g(z),$$

where $g(z) := \sum_{k=0}^{\infty} a_{p+k}(z - z_0)^k$ satisfies $g \in \mathcal{O}(\mathcal{U})$ and $g(z_0) = a_p \neq 0$.

Question 7.

a) If $z_0 \in Int(S)$, there exists $\varepsilon > 0$ such that $Ball_\varepsilon(0) \subseteq S$. Thus,

$$z_n = z_0 + \frac{1}{n} \in S$$

for n large enough and $z_n \to z_0$ as $n \to \infty$, i.e., z_0 is condensation point of S.

b) Let $z_0 \in S$ and $z_1 \in S$. As S is path-connected, there exists a path $\gamma : [0, 1] \to S$ with $\gamma(0) = z_0$ and $\gamma(1) = z_1$. Let

$$z_n := \gamma(1/n) \in S,$$

then $z_n \to \gamma(0) = z_0$ from the continuity of γ, i.e., z_0 is a condensation point of S.

Question 8.

b) Yes.

$$f(z) = \frac{\sin(2\pi/z)}{\sin(2\pi/i)}.$$

c) Applying Cauchy's theorem:

$$0 = \int_{|z|=1} \frac{f(z)dz}{(n+1)z - 1} = \frac{1}{n+1} \int_{|z|=1} \frac{f(z)dz}{z - \frac{1}{n+1}} = \frac{2\pi i}{n+1} f\left(\frac{1}{n+1}\right)$$

implies that $f(1/n + 1) = 0$ for all $n \geq 1$. Since f is continuous in $Ball_2(0)$, we must have $f(0) = 0$ and then applying the uniqueness theorem to get that $f = 0$.