HOMEWORK SHEET 8: SOLUTIONS

INTRODUCTION TO COMPLEX ANALYSIS FOR ELECTRIC ENGINEERING

Question 1.

a) The function $f(z) = \frac{e^{tz}}{z^2+1}$ is analytic in $\mathbb{C} \setminus \{\pm i\}$ and in particular in $Ball_{1/2}(0)$, therefore the integral is equal to 0.

b)(i) Yes. Let $a_n = \frac{1}{n^2}$, then it is easy to check that the radius of convergence is $R = 1$, while for every $|z| = 1$ we have $\sum |a_n z^n| = \sum \frac{1}{n^2}$ converges.

b)(ii) Yes. Let $a_n = 1$, then it is easy to check that the radius of convergence is $R = 1$, however for every $|z| = 1$ we have $|a_n z^n| = 1$ and so $\sum a_n z^n$ does not converge.

c) As $ord_{z_0}(f) = n$, there exists $g \in \mathcal{O}(\mathcal{U})$ such that $f(z) = (z - z_0)^n g(z)$ with $g(z_0) \neq 0$. Since g is continuous at z_0 and $g(z_0) \neq 0$, there exists $\varepsilon > 0$ such that the image of $g(z)$ on $Ball(z_0)$ lies in a domain of an analytic branch of log and we can define an analytic function $log(g(z))$ in $Ball(z_0)$. Therefore, let

$$h(z) := (z - z_0)e^{\frac{1}{n}log(g(z))}$$

which is an analytic function in $Ball_\varepsilon(z_0)$ such that $h(z)^n = (z - z_0)^n g(z) = f(z)$.

Question 2

a) We have

$$f\left(\frac{1}{n}\right) = \frac{1}{n^2+1} = \frac{1}{\frac{1}{\left(\frac{1}{n}\right)^2} + 1} = g\left(\frac{1}{n}\right),$$

where $g(z) = \frac{1}{\frac{1}{z^2}+1} = \frac{z^2}{z^2+1}$. As $f\left(\frac{1}{n}\right) = g\left(\frac{1}{n}\right)$ for n large enough (such that $\frac{1}{n} < \varepsilon$) and both f and g are analytic in $Ball_\varepsilon(0)$, the theorem of uniqueness guarantees that $f(z) = g(z)$ for all $z \in Ball_\varepsilon(0)$, in particular $f\left(\frac{\varepsilon}{2}\right) = g\left(\frac{\varepsilon}{2}\right) = \frac{\frac{\varepsilon^2}{4}}{\frac{\varepsilon^2}{4}+1} = \frac{\varepsilon^2}{\varepsilon^2+4}$.

b) Suppose that for every $n \in \mathbb{N}$ we have $f\left(\frac{1}{n}\right) = \frac{1}{n+1}$. Similarly to the previous part, let $g(z) = \frac{1}{\frac{1}{z}+1} = \frac{z}{z+1}$, thus $f\left(\frac{1}{n}\right) = g\left(\frac{1}{n}\right)$ for every $n \in \mathbb{N}$ and from the uniqueness theorem we get that $f(z) = g(z)$ for all $z \in Ball_{3/2}(0)$, however g is not defined at $z = -1 \in Ball_{3/2}(0)$, which is a contradiction.

c) If $f(\lambda z) = f(z)$ for all $z \in \mathbb{C}$, then we consider the Taylor expansions of both functions:

$$\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n (\lambda z)^n \Longrightarrow \sum_{n=0}^{\infty} a_n (1 - \lambda^n) z^n = 0$$

and that implies the coefficients must all vanish, i.e.,

$$a_n(1-\lambda^n) = 0, \ \forall n \geq 0.$$

As f is non-constant, we have $a_n \neq 0$ for some $n \geq 1$, hence $\lambda^n = 1$. Therefore, this is only possible when λ satisfies $\lambda^n = 1$ for some $n > 1$.

d) As $f(it) = f(i(t+1)) = f(i(t+\sqrt{2}))$ for all $t \in \mathbb{R}$, we know that $f(z) = f(0)$ for every

$$z \in X := \left\{ i(m\sqrt{2}+n) : n, m \in \mathbb{Z} \right\}.$$

So we have a set of points on which the function is constant; we would like to apply the theorem of uniqueness, for that we need the set to have a condensation point.

Consider the following subset of X, given by

$$X_1 := \left\{ i(m\sqrt{2} - [m\sqrt{2}]) : m \in \mathbb{Z} \right\}.$$

It is easy to see that

$$m_1\sqrt{2} + n_1 = m_2\sqrt{2} + n_2 \iff m_1 = m_2, n_1 = n_2,$$

therefore it means that the set X_1 is **infinite**; moreover, for every $m \in \mathbb{Z}$ we have $|i(m\sqrt{2} - [m\sqrt{2}])| \leq 1$, meaning that $X_1 \subseteq Ball_1(0)$ and so X_1 is **bounded**.

We have $f(z) = f(0)$ for every $z \in X_1$, which is infinite and bounded set, therefore the theorem of uniqueness guarantees that $f(z) = f(0)$ for every $z \in \mathbb{C}$, i.e., f is constant.

e) We know that the coefficients in the Taylor expansion is given by

$$c_{n,z_0} = \frac{f^{(n)}(z_0)}{n!},$$

therefore we get that $f^{(1001)}(z_0) = 0$ for all $z_0 \in \mathcal{U}$, which clearly implies that f must be a polynomial of degree at most 1000.

f) See question 6 in Tirgul 9.

Question 3.

a) Let $f(z) = z^4 - z$. By the maximum principle for $f(z)$ in $\overline{Ball_1(0)}$, we get

$$\max_{|z|\leq 1} |f(z)| = \max_{|z|=1} |f(z)| = \max_{|z|=1} |z^3 - 1|.$$

For every $|z| = 1$ we have

$$|z^3 - 1| = |e^{3i\theta} - 1| = \sqrt{(\cos(3\theta)-1)^2 + \sin(3\theta)^2} = \sqrt{2 - 2\cos(3\theta)}$$

which has a maximal value when $\cos(3\theta) = -1$, thus

$$\max_{|z|\leq 1} |f(z)| = 2.$$

Clearly, $\min_{|z|\leq 1} |f(z)| = 0$ as $|f(z)| \geq 0 = |f(0)|$.

b) As $\sin(z)$ is analytic and non-constant, from the maximum principle (and the fact that $\sin(z)$ has a period of 2π) we have

$$\max_{z\in[0,2\pi]^2} |\sin(z)| = \max \left\{ \max_{x\in[0,2\pi]} |\sin(x)|, \max_{x\in[0,2\pi]} |\sin(xi)|, \max_{x\in[0,2\pi]} |\sin(x+2\pi i)| \right\}.$$

Next,

$$\max_{x\in[0,2\pi]} |\sin(x)| = 1,$$

$$|\sin(xi)| = \left|\frac{e^{-x} - e^{x}}{2i}\right| = \frac{e^{x} - e^{-x}}{2}$$

is monotonic increasing (as its derivative is ≥ 0) and hence

$$\max_{x\in[0,2\pi]} |\sin(xi)| = \frac{e^{2\pi} - e^{-2\pi}}{2},$$

and

$$\begin{aligned}
|\sin(x+2\pi i)| &= |\sin(x)\cos(2\pi i) + \cos(x)\sin(2\pi i)| \\
&= \left|\sin(x)\left(\frac{e^{-2\pi}+e^{2\pi}}{2}\right) + \cos(x)\left(\frac{e^{-2\pi}-e^{2\pi}}{2i}\right)\right| \\
&= \sqrt{\sin^2(x)\left(\frac{e^{-4\pi}+2+e^{4\pi}}{4}\right) + \cos^2(x)\left(\frac{e^{-4\pi}-2+e^{4\pi}}{4}\right)} \\
&= \frac{1}{2}\sqrt{e^{-4\pi}+e^{4\pi}+2(\sin^2(x)-\cos^2(x))} = \frac{1}{2}\sqrt{e^{-4\pi}+e^{4\pi}-2\cos(2x)}
\end{aligned}$$

which implies that

$$\max_{x\in[0,2\pi]} |\sin(x+2\pi i)| = \frac{\sqrt{e^{-4\pi}+e^{4\pi}+2}}{2} = \frac{e^{-2\pi}+e^{2\pi}}{2}.$$

Therefore,

$$\max_{z\in[0,2\pi]^2} |\sin(z)| = \max\left\{1, \frac{e^{2\pi} \pm e^{-2\pi}}{2}\right\} = \frac{e^{2\pi}+e^{-2\pi}}{2}.$$

c)(i) We have $z\overline{z} = |z|^2 = 1$, then $\overline{z} = 1/z$ and

$$|z-a| = |\overline{z-a}| = \left|\frac{1}{z} - \overline{a}\right| = \frac{|1-\overline{a}z|}{|z|} = |1-\overline{a}z|.$$

c)(ii) Let

$$f(z) = \prod_{k=1}^{n} \frac{z-a_k}{1-\overline{a_k}z},$$

as $1-\overline{a_k}z = 0 \iff z = 1/\overline{a_k}$ and $|1/\overline{a_k}| > 1$, we get that f is analytic in $\overline{Ball_1(0)}$. From the first part of the question, for every $|z| = 1$ we have $|f(z)| = 1$ and as f is non-constant, we get from the maximum principle that for every $|z| < 1, |f(z)| < 1$.

e)(i) No. Any analytic f and z_0 such that $f(z_0) = 0$ is a counter example.

e)(ii) Yes. Suppose z_0 is a local minimum of $f(z)$. As f is continuous in a neighborhood of z_0 and $f(z_0) \neq 0$, it follows that there exists $\varepsilon > 0$ such that $Ball_\varepsilon(0) \subseteq \mathcal{U}$ and $f(z) \neq 0$ for every $z \in Ball_\varepsilon(0)$. Thus, the function $g(z) := 1/f(z)$ is (defined and) analytic in $Ball_\varepsilon(0)$ and it has a local maximum at z_0. Nevertheless, since f is non-constant so is g in $Ball_\varepsilon(0)$, which contradicts the maximum principle.

e)(iii) See question 2 in Tirgul 9.

f) Denote the Box by $[-a,a]^2$ for some $a>0$, the upper side $\ell_1=[-a+ai,a+ai]$, the right side $\ell_2=[a+ai,a-ai]$, the lower side $\ell_3=[a-ai,-a-ai]$ and the left side $\ell_4=[-a-ai,-a+ai]$. Consider the function

$$g(z):=\frac{f(z)+f(-z)+f(iz)+f(-iz)}{4}.$$

As $f\in\mathcal{O}(\overline{Box})$ it follows that $g\in\mathcal{O}(\overline{Box})$, since $z\in\overline{Box}$ implies that $-z,iz,-iz\in\overline{Box}$. Moreover, it is easily seen that

$$z\in\ell_1 \iff -z\in\ell_3 \iff iz\in\ell_4 \iff -iz\in\ell_2,$$

therefore for every $z\in\partial Box$, the points $z,-z,iz,-iz$ lie on different sides of the boundary of the box and thus $|g(z)|\le\frac{1}{4}(\ell_1+\ell_2+\ell_3+\ell_4)$. Applying the maximum principle for $g(z)$, we have

$$|f(0)|=|g(0)|\le\max_{z\in\partial Box}|g(z)|\le\frac{\ell_1+\ell_2+\ell_3+\ell_4}{4}.$$

Question 4.

a) As $ord_0(f)\ge n$, there exists $g\in\mathcal{O}(\overline{Ball_1(0)})$ such that

$$f(z)=z^n g(z),\ \forall z\in\overline{Ball_1(0)}.$$

Then for every $|z|=1$ we have $|g(z)|=|f(z)|\le 1$ and together with the maximum principle we get that $|g(z)|\le 1$ for every $z\in\overline{Ball_1(0)}$. Therefore,

$$|f(z)|=|z|^n|g(z)|\le|z|^n,\ \forall z\in\overline{Ball_1(0)}.$$

b) Applying Schwartz's Lemma, we know that $|f(z)|\le|z|$ for all $z\in Ball_1(0)$. Therefore, $|f(z^n)|\le|z|^n<r^n$ for every $z\in Ball_r(0)$, while $\sum_{n=0}^{\infty}r^n$ converges and so $\sum_{n=0}^{\infty}f(z^n)$ converges uniformly in $Ball_r(0)$, for every $0<r<1$.

c) See question 5 in Tirgul 9.

d) Let $g(z)=\frac{1}{2}(f(z)+f(-z))$. As $f\in\mathcal{O}(\overline{Ball_1(0)})$ and $|f(z)|\le 1$ for $|z|\le 1$, we get that $g\in\mathcal{O}(\overline{Ball_1(0)})$, with the property that $|g(z)|\le 1$ for $|z|\le 1$. Moreover, $g(0)=0$ and $g'(0)=0$, so $ord_0(g)\ge 2$ and applying the first part of the question to obtain that $|g(z)|\le|z|^2$, therefore $|f(z)+f(-z)|\le 2|z|^2$ in $\overline{Ball_1(0)}$.