Selected Answers to HW 7

Question 1

By Cauchy's formula:

$$|p_j| = \left| \frac{1}{2\pi i} \int_{|z|=1} \frac{p(z)}{z^{n+1}} dz \right| \leq \max_{|z|=1} |p(z)| \leq 1.$$

where the first inequality is by the integral triangle inequality, and the fact that the length of the circle is 2π.

Question 3

Item c

This is immediately by Cauchy's formula. If $|f(z)| \leq C$ on the circle, then

$$|a_n| = \left| \frac{1}{2\pi i} \int_{|z|=\rho} \frac{f(z)}{z^{n+1}} dz \right| \leq \rho 2\pi \max_{|z|=\rho} |f(z)| \, \rho^{-(n+1)} \leq \frac{C}{\rho^n}.$$

Item d

Fix $q = e^{2\pi i \alpha}$ for a non-rational α. Obviously any constant function has that $f(z) = f(qz)$. We will show that any function with this property *must* be constant. Suppose that $f(z)$ is defined on the ring $r < |z| < R$, then we can write $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$. Multiplying by q doesn't change the norm of z thus $f(qz)$ is defined in the same ring. Thus

$$f(qz) = \sum_{n=-\infty}^{\infty} (a_n q^n) z^n.$$

In particular by uniqueness of the Laurent series, $a_n = a_n q^n$. Since α is not rational, then $n\alpha$ is never an integer when $n \neq 0$. thus $q^n = e^{2\pi i (n\alpha)}$ is never 1. This indicates that for all $n \neq 0$, $a_n = 0$. Thus f is constant.

Another Solution q is not rational, which means that $q^t \neq 1$ for any integer $t \neq 0$. This implies that for any $n \neq m$ $q^n \neq q^m$ (because $q^{n-m} \neq 1$). Thus the sequence $w_n = q^n z_0$ is a sequence of infinitely many distinct points for any z_0. For any w_n we note that $f(w_n) = f(w_0)$.

This sequence lies in the circle of radius $r = |z_0|$ thus it has a convergent sub-sequence. From the uniqueness theorem, f has a sequence that has a sequence with a cluster point on which it is constant - the f is constant.

Question 4

Item a

See similar questions in Tirgul 9.

Item b

When f has an essential point at z_0 and g has a pole of order m:

i. $f+g$ has an essential point, since the Laurent series has infinite number of negative terms.

ii. $f(z)g(z)$ - an essential point. Note that if $g(z)$ is a pole then $g(z) = \frac{h(z)}{(z-z_0)^N}$ for some analytic function $h(z)$ so that $h(z_0) \neq 0$. $\lim_{z\to z_0} f(z)h(z)$ doesn't exist (otherwise $\lim_{z\to z_0} f(z)$ would exist, thus the Laurent series of $f(z)h(z)$ has infinitely many negative terms. Hence the Laurent expansion of $\frac{f(z)h(z)}{(z-z_0)^N} = f(z)g(z)$ has infinitely many negative terms - and z_0 is an essential point[1].

iii. $1/f$ has an essential point, since if there was a limit to $1/f(z)$ then there was a limit to $f(z)$ when $z \to z_0$ (which might have been infinity).

iv. f' has an essential point, since the a derivative of its essential part can be taken one-by-one, i.e.

$$\left(\sum_{n=-\infty}^{-1} a_n(z-z_0)^n\right)' = \sum_{n=-\infty}^{-1} na_n(z-z_0)^{n-1}.$$

so if the series of $f(z)$ has infinite number of negative terms, then so does the series for $f'(z)$.

v. $g^{(n)}$ By the same reasoning as above g has a pole of order $m+n$.

vi. $e^{f(z)}$ is an essential point. The range of the function f is dense in $\mathbb{C}$ in any ball around z_0, thus there are two sequences where $z_n \to z_0$,$w_n \to z_0$ where $f(z_n) \to 0$ and $f(w_n) \to 1$. In particular by continuity we get that $e^{f(z_n)} \to 1$ and $e^{f(w_n)} \to e$, thus no limit exists when $z \to z_0$.

Item c

Notice that we can extend $g(z) = (z-z_0)f(z)$ analyticly to z_0, and this function has a zero at z_0. Thus by a theorem we saw in class, we can write $g(z) = z^k h(z)$ for $k > 0$ and an analytic function $h(z)$ in a ball around z_0. In particular we can write $f(z) = z^{k-1}h(z)$ and thus z_0 is a removable singularity for f.

Item d

It is enough to prove that 0 is an essential singular point for the function $f(z) = Log(1+z^3)e^{\frac{1}{z}} - cos^2(z)$. Indeed, $cos^2(z)$ has no singularity at 0 and thus it is enough to show that $Log(1+z^3)e^{\frac{1}{z}}$ has an essential singularity. Furthermore, $Log(1+z^3) = z^3 g(z)$ for some $g(z)$ analytic in a ball around 0 s.t. $g(0) \neq 0$. The function $e^{\frac{1}{z}}g(z)$ has an essential singularity since if it had a limit (including infinity) then so would $e^{\frac{1}{z}}$. If $e^{\frac{1}{z}}g(z)$ has an essential singularity point at 0 then so does $Log(1+z^3)e^{\frac{1}{z}} = z^3 e^{\frac{1}{z}} g(z)$ since multiplying by z^3 doesn't change the fact that the Laurent series doesn't end at some finite a_n.

Item e

Note that from the fact that $|f(z)| \geq Ce^{\frac{1}{|z|}}$ for some $C > 0$ we get that $\lim_{z\to 0}|f(z)| = \infty$. Thus $f(z)$ has a pole of order N at 0 for some $N \geq 1$. Hence,

$$f(z) = \frac{g(z)}{z^N}$$

for some $g(z)$ analytic in some ball around 0. Thus

$$|f(z)| \geq Ce^{\frac{1}{|z|}} \Rightarrow |g(z)| \geq C|z^N|e^{1/|z|}$$

[1] It is tempting to say that because $f(z)$ Laurent series has infinitely many negative terms then so does the Laurent series of $f(z)g(z)$ directly. However, this might not be true when $f(z)$ is not defined in a ball around z_0. For example take $f(z) = \sum_{n=1}^{\infty} z^{-n}$ and $g(z) = 1 - \frac{1}{z}$.

But the expression $|z^N|e^{1/|z|}$ still goes to infinity for every N as $|z| \to 0^+$. This is a contradiction to the fact that g is bounded around 0.

Question 5

Item a

$$\hat{h}(n) = \frac{1}{2\pi i}\int_0^{2\pi} h(e^it)ie^{it}e^{-int}dt = \frac{1}{2\pi i}\int_{\partial B_1(0)} \frac{g(z)}{z^{n+1}}dz.$$

Notice that as $\frac{g(\xi)}{z-\xi}$ is continuously diffrentiable (by t), then

$$\frac{\partial^n}{dz^n} f_1(z) = \frac{1}{2\pi i}\int_{\partial B_1(0)}$$

$$\frac{\partial^n}{dz^n}\frac{g(\xi)}{z-\xi}d\xi = \frac{n!}{2\pi i}\int_{\partial B_1(0)} \frac{g(\xi)}{(z-\xi)^{n+1}}d\xi$$

So in particular, for all $n \geq 0$

$$\hat{h}(n) = \frac{1}{n!}f_1^{(n)}(0) = a_n,$$

where a_n is the n-th Taylor expansion coefficient.

Similarly we can prove that for all $n < 0$

$$\hat{h}(n) = \frac{1}{n!}f_2^{(n)}(0) = b_n.$$

which is the n-th Laurent coefficient.

Item b

First note that because g is analytic in a ring $1-\varepsilon < |z| < 1+\varepsilon$ then for any $|w| < 1$

$$\int_{\partial B_1(0)} \frac{g(\xi)}{\xi - w}d\xi = \int_{\partial B_{1+\varepsilon/2}(0)} \frac{g(\xi)}{\xi - w}d\xi.$$

Thus we can define $f_1(w$ by the integral on $|\xi| = 1 + \frac{\varepsilon}{2}$ and use the same definition to define f_1 on $B_{1+\varepsilon/2}(0)$.

We do a similar thing for f_2 (but with $1 - \frac{\varepsilon}{2}$).

Thus by Cauchy's integral formula (where we use the ring $\left\{1 - \frac{\varepsilon}{2} < |z| < 1 + \frac{\varepsilon}{2}\right\}$ as our domain):

$$h(t) = g(e^{it}) = \int_{\partial\left\{1-\frac{\varepsilon}{2}<|z|<1+\frac{\varepsilon}{2}\right\}} \frac{g(\xi)}{\xi - e^{it}}d\xi = f_1(e^{it}) + f_2(e^{it}).$$