Functions of several complex variables and their critical points

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Homework 1. Submission date: 19.11.2020

Questions to submit: 1.c, 2.b, 2.d, 3.c, 4.a.

Notations/conventions: $\mathcal{U} \subseteq \mathbb{C}^n$ is open, $\underline{z} = (z_1, \dots, z_n)$, $\underline{m} = (m_1, \dots, {}_n m_n)$, $|\underline{m}| = \sum m_j$.

(1) (a) Prove: f is $\mathbb{C}$-differentiable at $\underline{z}_0 \in \mathbb{C}^n$ iff $f(\underline{z}) = f(\underline{z}_0) + \sum\limits_{j=1} R_j(z)(z_j - z_{0,j})$ holds near $\underline{z}_0$. Here $\{R_j(\underline{z})\}$ are some continuous functions.

(b) Prove: if f is $\mathbb{C}$-differentiable at $\underline{z}_0 \in \mathbb{C}^n$ then the derivative $f'|_{\underline{z}_0}$ is defined uniquely.

(c) Consider $\mathbb{C}^n \supseteq \mathcal{U} \stackrel{f}{\to} \mathbb{C}$ as a real valued function, $\mathbb{R}^{2n} \supseteq \mathcal{U} \stackrel{[f]}{\to} \mathbb{R}^2$, with $z_j = x_j + i \cdot y_j$. Define (formally): $\frac{\partial [f]}{\partial z_j} := \frac{\partial_{x_j}[f] - \partial_{y_j}[i \cdot f]}{2}$ and $\frac{\partial [f]}{\partial \bar{z}_j} := \frac{\partial_{x_j}[f] + \partial_{y_j}[i \cdot f]}{2}$.
Prove: f is $\mathbb{C}$-differentiable at $\underline{z}_0$ iff $[f]$ is $\mathbb{R}$-differentiable and $\{\frac{\partial [f]}{\partial \bar{z}_j} = 0\}_{j=1,\dots,n}$.
(These are $2n$ real conditions on the two real functions, $Re(f)$, $Im(f)$. Thus, for $n > 1$, this system is overdetermined.)

(d) Prove: if $f \in \mathcal{O}(\mathcal{U})$ then it is infinitely $\mathbb{C}$-differentiable, and

$$\partial^{\underline{m}} f(z) := \partial_{z_1}^{m_1} \cdots \partial_{z_n}^{m_n} f(z) = \frac{m_1! \cdots m_n!}{(2\pi i)^n} \int_{Torus_r(z)} \frac{f(\xi) d\xi_1 \cdots d\xi_n}{(\xi_1 - z_1)^{m_1+1} \cdots (\xi_n - z_n)^{m_n+1}}.$$

(e) Verify: $\mathcal{O}(\mathcal{U})$, $C^\omega(\mathcal{U})$, $C^\infty(\mathcal{U})$ are (commutative) $\mathbb{C}$, $\mathbb{R}$-algebras.

(2) (a) Suppose $\sum a_{\underline{m}} \underline{z}^{\underline{m}}$ converges uniformly on $\mathcal{U}$. Prove: $\partial_{z_j} \sum a_{\underline{m}} \underline{z}^{\underline{m}} = \sum a_{\underline{m}} \partial_{z_j}(\underline{z}^{\underline{m}})$ and $\int (\sum a_{\underline{m}} \underline{z}^{\underline{m}}) dz_1 \dots dz_n = \sum a_{\underline{m}} (\int \underline{z}^{\underline{m}} dz_1 \dots dz_n)$.

(b) We have proven Abel's theorem for the power series $\sum a_{\underline{m}} \underline{z}^{\underline{m}}$. Strengthen the statement to: "If for some $\underline{w} \in \mathbb{C}^n$ the set $\left\{|a_{\underline{m}} \underline{w}^{\underline{m}}|\right\}_{\underline{m}}$ is 'sub-exponentially' bounded, i.e. $\lim\limits_{|\underline{m}| \to \infty} \frac{ln(1 + |a_{\underline{m}} \underline{w}^{\underline{m}}|)}{|\underline{m}|} = 0$, then …".

(c) Define the *set of convergence* of a series, $S := \{\underline{z} | \ \sum a_{\underline{m}} \underline{z}^{\underline{m}} \text{ converges}\} \subseteq \mathbb{C}^n$.
  (i) Recall that for $n = 1$ one has $Disc_R(0) \subseteq S \subseteq \overline{Disc_R(0)}$, where R is the radius of convergence. Does this hold for $n > 1$ and polydiscs?
  (ii) Disprove: $S \subseteq \overline{Int(S)}$; S is path-connected
(Hint: consider $f(z_1, z_2) = \frac{z_1}{1 - z_2}$, $f(z_1, z_2) = \frac{1}{1 - z_1 z_2}$)

(d) Find the maximal $r_1, \dots, r_n$ for which the Taylor series of $f(z) = \prod_{j=1}^n tan \frac{z_j}{c_j}$ converges in the polydisc $\prod Disc_{r_j}(0)$.

(3) Let $f, g \in \mathcal{O}(\mathcal{U})$, with $\mathcal{U}$ path-connected.

(a) Prove the uniqueness theorem: if $f = g$ in a neighborhood of a point $\underline{z}_0 \in \mathcal{U}$ then $f = g$ on $\mathcal{U}$.

(b) For $n = 1$ the uniqueness theorem is stronger, "If $f = g$ on a sequence of points that converges inside $\mathcal{U}$, then …". Give an example of $f, g \in \mathcal{O}(\mathbb{C}^n)$ such that $f(z_1, \dots, z_{n-1}, 0) = g(z_1, \dots, z_{n-1}, 0)$, but $f \neq g$.

(c) Let $\mathbb{R}^n \subset \mathbb{C}^n$ be the real part, defined by $\{Im(z_j) = 0\}_j$. Prove: if $f = g$ in a real neighborhood of a point (i.e. in some $Ball_\epsilon(\underline{z}_0) \cap \mathbb{R}^n$) then $f = g$ on $\mathcal{U}$.

(4) (a) Prove the Weierstrass theorem: if a sequence of functions $\{f_k \in \mathcal{O}(\mathcal{U})\}$ converges uniformly to f then f is holomorphic. (If you have to change the order of integration/summation/limit in the proof, justify why is this possible)

(b) Prove Liouville's theorem: if $f \in \mathcal{O}(\mathbb{C}^n)$ is bounded, then it is constant.