Functions of several complex variables and their critical points

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Questions to submit: 2.c, 2.d, 3.a, 3.b, 3.e, 3.f, 4.b, 4.d.

Below $\mathbb{k}$ is one of $\mathbb{R}$, $\mathbb{C}$, and R is one of $C^\infty(\mathbb{R}^n,0)$, $\mathbb{k}[[\underline{x}]]$, $\mathbb{k}\{\underline{x}\}$.

1. One step in the proof of IFT was: if the derivative $\partial_{\underline{w}} \underline{f}(\underline{z},\underline{w})|_{(z_0,w_0)} \in Mat_{m\times m}(\mathbb{C})$ is non-degenerate, then so is the $2m \times 2m$ matrix composed of the real and imaginary parts. This is also implied by any of the following statements, prove them.

   Given a linear map of $\mathbb{K}$-vector spaces $V \stackrel{L}{\to} W$, and a subfield $\mathbb{k} \subset \mathbb{K}$. Consider V, W as $\mathbb{k}$-vector spaces, and take the induced ($\mathbb{k}$-linear) map $V_{\mathbb{k}} \stackrel{[L]_{\mathbb{k}}}{\to} W_{\mathbb{k}}$.
   (a) L is surjective iff $[L]_{\mathbb{k}}$ is surjective.
   (b) L is injective iff $[L]_{\mathbb{k}}$ is injective.

2. (a) Verify that the equivalence used to define the germs (of sets/of functions) is indeed an equivalence relation.
   (b) Verify the basic set-theoretic identities, for a finite number of germs: $\cap_i (X_i, o) = (\cap_i X_i, o)$, $\cup_i (X,o) = (\cup_i X_i, o)$, $(X,o) \setminus (Y,o) = (X \setminus Y, o)$, $\prod_i (X_i, 0) = (\prod_i X_i, o)$.
   (c) Show that the notions "a closed/open subgerm" are well defined. We will see later that for analytic set-germs the notions of "path-connected/contractible/simply-connected" are well defined.
   (d) Denoting by $[f]_p$ the germ of f at a point p, verify that the basic operations (on a finite number of function germs) are well defined and satisfy: $\sum_i [f_i]_p = [\sum_i f_i]_p$, $\prod_i [f_i]_p = [\prod_i f_i]_p$.
   Verify: with these operations the sets $C^\infty(\mathbb{R}^n, z_0)$, $C^\omega(\mathbb{R}^n, z_0)$, $\mathcal{O}(\mathbb{C}^n, z_0)$ become local $\mathbb{k}$-algebras.

3. (a) Prove: the rings $\mathbb{k}[[x]]$, $\mathbb{k}\{x\}$ (for one variable) are principal ideal domains. What about $C^\infty(\mathbb{R}^1, o)$?
   (b) Verify: IFT holds in the rings $C^\infty(\mathbb{R}^n, 0)$, $\mathbb{k}\{\underline{z}\}$. (How to obtain the $\mathbb{R}\{\underline{x}\}$-version from the $\mathbb{C}\{\underline{z}\}$-version?)
   (c) Prove the IFT for $\mathbb{k}[[\underline{x}]]$: if $f(\underline{z},\underline{w}) = 0$ are power series equations, with $f(0,0) = 0$, and $\partial_{\underline{w}} f|_{(0,0)}$ non-degenerate, then ...
   (d) For $f \in \mathfrak{m} \subset R$ prove: $\sqrt[p]{1+f} \in R$.
   (e) Prove: $\mathcal{O}(\mathcal{U}) \stackrel{localization}{\to} \mathcal{O}(\mathcal{U})_{\mathfrak{m}_{z_0}} \stackrel{\phi}{\hookrightarrow} \mathcal{O}(\mathbb{C}^n, z_0) \stackrel{z\to z-z_0}{\stackrel{\sim}{\to}} \mathcal{O}(\mathbb{C}^n, o) \stackrel{\psi}{\stackrel{\sim}{\to}} \mathbb{C}\{\underline{z}\}$ and
   $C^\omega(\mathcal{U}) \stackrel{localization}{\to} C^\omega(\mathcal{U})_{\mathfrak{m}_{z_0}} \stackrel{\phi}{\hookrightarrow} C^\omega(\mathbb{R}^n, z_0) \stackrel{z\to z-z_0}{\stackrel{\sim}{\to}} C^\omega(\mathbb{R}^n, o) \stackrel{\psi}{\stackrel{\sim}{\to}} \mathbb{R}\{\underline{z}\}$.
   Show that ϕ is non-surjective.
   (f) Prove: $C^\infty(\mathbb{R}^n) \stackrel{localiz}{\to} C^\infty(\mathbb{R}^n)_{\mathfrak{m}_{z_0}} \stackrel{\phi}{\stackrel{\sim}{\to}} C^\infty(\mathbb{R}^n, z_0) \stackrel{z\to z-z_0}{\stackrel{\sim}{\to}} C^\infty(\mathbb{R}^n, o)$ with $C^\infty(\mathbb{R}^n, z_0) \stackrel{\sim}{\to} C^\infty(\mathcal{U}) / \sum_{\epsilon>0} I_{Ball_\epsilon(z_0)}$

4. (a) For $f \in R$ prove: $Jac(f) = \mathfrak{m}$ iff o is a non-degenerate critical point of f.
   (b) Let $g \in \mathfrak{m} \subset R$. Construct a (explicit) change of coordinates that brings $x^2 + y^2 + xy \cdot g$ to $x^2 + y^2$.
   (c) Prove the generalized Morse lemma: if $f'|_o = 0$ and $rank(f^{(2)}|_{z_0}) = r < n$ then in some local coordinates $f(\underline{z}) = \sum\limits_{i=1}^{r} (\pm) z_i^2 + f_{\geq 3}(z_{r+1}, \dots, z_n)$, where $f_{\geq 3} \in \mathfrak{m}^3$.
   (d) Let $f \in \mathfrak{m} \subset R$, for $R = \mathbb{k}\{\underline{z}\}$ or $C^\infty(\mathbb{R}^n, o)$. Assume $det[f''|_o] \neq 0$. Prove:
      (i) The germ $f^{-1}(0) \subset (\mathbb{k}^n, o)$ is contractible. (Check both the case $f'|_o = 0$ and $f'|_o \neq 0$.)
      (ii) For $\mathbb{k} = \mathbb{C}$ there exists a neighborhood $o \in \mathcal{U}$ such that for any $0 < \epsilon \ll 1$ the subset $f^{-1}(\epsilon) \cap \mathcal{U}$ is homotopy equivalent to $S^{n-1} \subset \mathcal{U} \subset \mathbb{C}^n$. (This is the simplest example of the *Milnor fibre.*) What happens for $\mathbb{k} = \mathbb{R}$?

5. Verify the basic properties of $ord(f)$:
   (a) $ord(f) = p$ iff $\big[f(o) = f^{(1)}|_o = \cdots = f^{(p-1)}|_o = 0$ and $f^{(p)}|_o \neq 0\big]$.
   (b) Describe all the elements $f \in R$ for which $ord(f) = \infty$.
   (c) $ord(f \cdot g) = ord(f) ord(g)$, $ord\frac{f}{g} = ord(f) - ord(g)$, $ord(f \pm g) \geq min(ord(f), ord(g))$.
   (d) If $(f) = (g) \subset R$ then $ord(f) = ord(g)$.