Functions of several complex variables and their critical points

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Homework 3. Submission date: 03.12.2020

Questions to submit: 1.b.iv, 3.b, 4.b, 5.c, 6.b, 6.c, 6.d.

Below $\Bbbk$ is one of $\mathbb{R}$, $\mathbb{C}$, and R is one of $C^\infty(\mathbb{R}^n,0)$, $\Bbbk[[\underline{x}]]$, $\Bbbk\{\underline{x}\}$.

(1) (a) (i) Which of the following ideals (in R) is radical? For those that are not radical, compute their radicals. i. $\left(\frac{sin(xy)\cdot tan(yz)\cdot ln(1+xz)}{sin(xyz)}\right)$ ii. (x^m-y^n, y^m+x^n) iii. $\mathfrak{m}^\infty\subset C^\infty(\mathbb{R}^n,0)$.

(ii) Prove: $(x^3,y^2)=(x^3+y^4-y^7, y^2+x(\sqrt{1+x^2}-1)$.

(b) (i) Which of the following rings are Noetherian?
i. $C^\infty(\mathbb{R}^n,0)$ ii. $\Bbbk[x_1,x_2,\dots]$ iii. ${}^{S}\!/_{I}$ for S-Noetherian.
iv. Localization of a Noetherian ring. v. $C^\omega(\mathcal{U})$, $\mathcal{O}(\mathcal{U})$ for an open domain $\mathcal{U}\subseteq\Bbbk^n$.

(ii) Suppose S is Noetherian. Prove: for any system of generators $\{f_\alpha\}_\alpha$ of an ideal $I\subset S$ there exists a finite generating subsystem. (How does this differ from the definition?)

(2) (a) Prove: a locally analytic subset in $\mathcal{U}\subset\mathbb{C}^n$ is analytic iff it is closed.
(b) For an analytic subset $X\subsetneq\mathcal{U}$ prove: X is nowhere dense in $\mathcal{U}$.

(3) (a) All the ideals below are taken in the ring $\mathbb{C}\{\underline{x}\}$. (Dis)Prove:
(i) If $I_1\subset I_2$ then $V(I_1)\supsetneq V(I_2)$.
(ii) If $(X_1,o)\subsetneq(X_2,o)$ then $I(X_1)\supsetneq I(X_2)$. (Here (X_i,o) are arbitrary germs, not necessarily analytic. Does this hold for analytic germs?)
(iii) $V(I(X,o))=(X,o)$. $I(V(I))\supseteq\sqrt{I}$.
(b) Show (by examples) that the classical Nullstellensatz does not hold in the rings $\mathbb{R}[\underline{x}]$,$\mathbb{R}\{\underline{x}\}$,$C^\infty(\mathbb{R}^n,0)$.

(4) (a) For $f\in\Bbbk[[\underline{x}]]$ prove: there exists $\phi\in GL(n,\Bbbk)\circlearrowright\Bbbk^n$ such that ϕ^*f is z_n-regular. (Verify that your proof holds for $\Bbbk$ - any infinite field.)
(b) Let $\{f_i\}$ be at most countable collection in $\Bbbk[[\underline{x}]]$, for $\Bbbk\in\mathbb{R},\mathbb{C}$. Prove: there exists $\phi\in GL(n,\Bbbk)\circlearrowright\Bbbk^n$ such that all $\{\phi^*f_i\}$ are z_n-regular. (You can use the fact: the zero set of a polynomial is nowhere dense in $\Bbbk^n$.)
(c) Prove the uniqueness in the Weierstraß preparation theorem.
(d) Prove the Weierstraß division theorem for the ring $\Bbbk[[\underline{x}]]$.

(5) (a) Prove: i. If R is a domain then $R[[\underline{x}]]$ is a domain. ii. Any subring of a domain is a domain.
(b) Show that $C^\infty(\mathbb{R}^p,0)$ is not a domain.
(c) Suppose the curve germ $V(f)\subset(\mathbb{C}^2,o)$ is not smooth. Prove: ${}^{\Bbbk\{x,y\}}\!/_{(f)}$ is not a unique factorization domain. (Hint: you can assume f is irreducible and in the Weierstraß form.)
(d) Verify: the analytic germ $(X,o)\subset(\mathbb{C}^n,o)$ is irreducible iff $I(X,o)\subset\mathbb{C}\{\underline{x}\}$ is a prime ideal iff ${}^{\Bbbk\{\underline{x}\}}\!/_{I(X,o)}$ is a domain.
(e) Describe the decomposition of $V(x^p-y^p,\ y^q-z^q)\subset\mathbb{C}^3$ into irreducible components.

(6) (a) Let $A\in Mat_{m\times n}(R)$. Prove: the function germ $(\Bbbk^n,o)\stackrel{rank(A)}{\to}\mathbb{N}$ is locally non-decreasing.
(b) Prove: if the point $x_o\in X$ is regular then (X,x_0) is the germ of a complex manifold. ("Smoothness is an open property")
(c) Let $f\in\mathcal{O}(\mathbb{C}^n,o)$ be square-free. Prove: $Sing(V(f))\subsetneq V(f)$. (Hint: assume $f'|_{V(f)}=0$, use the Nullstellensatz, pass to $f^{(2)}|_{V(f)}$, and iterate.) Conclude: $Smooth(V(f))\subset V(f)$ is open and $\overline{Smooth(V(f))}=V(f)$.
(d) Generalize (c) to the arbitrary analytic subset $X\subset\mathbb{C}^n$.