Functions of several complex variables and their critical points

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Questions to submit: 2.a, 2.d, 3.c, 3.e, 5.a, 5.d, 5.f.

Below $\Bbbk$ is one of $\mathbb{R}$, $\mathbb{C}$, and R is one of $C^\infty(\mathbb{R}^n,0)$, $\Bbbk[[\underline{x}]]$, $\Bbbk\{\underline{x}\}$.

(1) (Be careful with the germs) Define $\mathbb{C}^2 \supset \mathcal{U} \stackrel{\pi}{\to} \mathbb{C}^2$ by $(x,y) \to (xy, x)$.
Does $\pi(\mathcal{U}, pt) = (\pi(\mathcal{U}), \pi(pt))$ hold for every $pt \in \mathcal{U}$?

(2) Let $f \in \mathbb{C}\{\underline{x}\}$. The critical locus of the function, $Crit(f) \subset (\mathbb{C}^n, o)$, is defined by the ideal $Jac(f) \subset R$. The singular locus of the hypersurface, $Sing(V(f)) \subset (\mathbb{C}^n, o)$, is defined by the ideal $(f) + Jac(f) \subset R$, called the Tjurina ideal.

- (a) Suppose $ord_o(f) = p$, present $f = f_p + f_{>p}$. Suppose $o \in V(f_p) \subset (\mathbb{C}^n, o)$ is an isolated singular point.
  - Prove: the images of $\partial_1 f, \dots, \partial_n f$ in ${}^{\mathfrak{m}^p}/_{\mathfrak{m}^{p+1}}$ are linearly independent.
  - Determine the necessary and sufficient condition on p, n to ensure $Jac(f) = \mathfrak{m}^{p-1}$.
- (b) Prove: $o \in \mathbb{C}^n$ is an isolated critical point of f iff $\sqrt{Jac(f)} = \mathfrak{m}$.
- (c) Verify: $Sing(V(f), o) = (Sing(V(f)), o)$.
- (d) Suppose $f(o) = 0$, prove $(Crit(f), o) = Sing(V(f), o) \subset (\mathbb{C}^n, o)$, as sets. (Because of this one often calls $Sing(V(f), o)$ "the singular locus of f".) You can use q.6 of hwk.3.
  Conclude: $o \in V(f)$ is an isolated singularity of $V(f)$ iff o is an isolated critical point of f.

(3)
- (a) Are the rings $C^\infty(\mathbb{R}^n,0)$, $\Bbbk[[\underline{x}]]$, $\Bbbk\{\underline{x}\}$ closed under compositions? (Give the precise statement)
- (b) For an analytic germ $(X, x_o) \subset (\mathbb{C}^n, x_o)$ verify: $\mathcal{O}(X, x_o)$ is a local Noetherian, reduced $\mathcal{O}(\mathbb{C}^n, x_o)$-algebra. Moreover, (X, x_o) is smooth iff $\mathcal{O}(X, x_o) \approx \mathbb{C}\{\underline{z}\}$.
- (c) For a homomorphism of analytic algebras $\mathcal{O}(Y, y_o) \stackrel{\phi}{\to} \mathcal{O}(X, x_o)$ verify:
  - $\phi(\mathfrak{m}_{\mathcal{O}(Y,y_o)}) \subseteq \mathfrak{m}_{\mathcal{O}(X,x_o)}$. (What is the geometric meaning?)
  - $\phi(\sum f_i) = \sum \phi(f_i)$, for any $\sum f_i \in \mathcal{O}(Y, y_o)$. (Here the sum is possibly infinite.)
- (d) Fix some elements $f_1, \dots, f_n \in \mathfrak{m} \subset \mathcal{O}(X, x_o)$. Prove: there exists (and unique) homomorphism $\mathbb{C}\{\underline{z}\} \stackrel{\phi}{\to} \mathcal{O}(X, x_o)$ satisfying $\phi(z_i) = f_i$.
- (e) Denote by $Hom(\mathcal{O}(Y, y_o), \mathcal{O}(X, x_o))$ the set of homomorphisms of analytic algebras. Show that $Hom(\mathcal{O}(Y, y_o), \mathcal{O}(X, x_o))$ is not additive.
- (f) Verify the details of the correspondence statement $Maps((X,o),(Y,o)) \rightleftarrows Hom_{alg}(\mathcal{O}(Y,o), \mathcal{O}(X,o))$. (This might be tedious, but it should be done.)

(4)
- (a) Verify: if a homomorphism $R \stackrel{\phi}{\to} S$ is finite/injective then ${}^{R}/_{\phi^{-1}(I)} \to {}^{S}/_{I}$ is finite/injective.
  Verify: the composition of finite homomorphisms is finite. (Give the geometric interpretation)
- (b) Verify: the embedding of analytic germs is a finite morphism. (What is the corresponding algebraic statement?)

(5)
- (a) Compute the codimension of the set $V(xz - y^2,\ x^3 - z^5,\ y^3 - z^4) \subset \mathbb{C}^3$.
- (b) Verify: $dim\mathcal{O}(X, x_o) = min\{d |$ exists a finite morphism $\mathcal{O}(\mathbb{C}^d, o) \to \mathcal{O}(X, x_o)\}$.
  (Give the geometric interpretation)
- (c) For the decomposition $(X, x_o) = \cup(X_i, x_o)$ verify: $dim(X, x_o) = \max_i\{dim(X_i, x_o)\}$.
- (d) Prove: $codim(V(f)) = 1$ for $0 \neq f \in \mathfrak{m} \subset \mathbb{C}\{\underline{z}\}$.
- (e) Prove: if $0 \neq f \in \mathfrak{m} \subset \mathcal{O}(X, x_o)$ then $dim\Big((X, x_o) \cap V(f)\Big) \geq dim(X, x_o) - 1$.
  Moreover, if f is not a zero divisor on $\mathcal{O}(X, x_o)$ then $dim\Big((X, x_o) \cap V(f)\Big) = dim(X, x_o) - 1$.
- (f) Prove: if $(X, x_o) \subset (\mathbb{C}^n, x_o)$ is an irreducible germ of dimension $n-1$ then $(X, x_o) = V(f)$.
- (g) Let $(X, x_o) \subseteq (Y, x_o)$, with (X, x_o) irreducible and $\dim(X, x_o) = \dim(Y, x_o)$. Prove: (X, x_o) contains an ireducible component of (Y, y_o).
- (h) For $\mathfrak{m} \subset \mathcal{O}(X, x_o)$ verify: $edim(X, x_o) = dim_{\mathbb{C}} {}^{\mathfrak{m}}/_{\mathfrak{m}^2}$. (In particular the $edim$ does not depend on the embedding.) Verify: $edim(X, x_o) = dim(X, x_o)$ iff (X, x_o) is smooth.