Functions of several complex variables and their critical points

(201.2.0441. Fall 2020. Dmitry Kerner)

Homework 5. Submission date: 20.12.2020.

Questions to submit: 1.a, 1.d, 1.e, 1.f. 2.a.i. 3.a.i. 3.c.

Below $\Bbbk$ is one of $\mathbb{R}$, $\mathbb{C}$, and R is one of $C^\infty(\mathbb{R}^n,0)$, $\Bbbk[[\underline{x}]]$, $\Bbbk\{\underline{x}\}$.

(1) We have defined the group actions $\mathcal{R},\mathcal{K} \circlearrowright R$. (Verify that these are indeed group-actions.)

- (a) Suppose $f(o)=0$ and $f'|_o=0$. Write down the transformation rule for the matrix $f^{(2)}|_o$ under the $\mathcal{R}$, $\mathcal{K}$-actions. Verify: $rank(f^{(2)}|_o)$ is $\mathcal{K}$-invariant.
  For $\Bbbk=\mathbb{R}$ denote by (n_+,n_0,n_-) the number of the positive/zero/negative eigenvalues of $f^{(2)}|_o$.
  Verify: the triple (n_+,n_0,n_-) is preserved under $\mathcal{R}$-equivalence.
  Verify: the numbers $n_0, |n_+-n_-|$ are preserved under $\mathcal{K}$-equivalence.
- (b) Define $\mathcal{R}^{(j)} := \{g\in\mathcal{R}|\ g(\underline{x})-\underline{x}\in\mathfrak{m}^j\cdot R^{\oplus n}\}$. Prove: $\mathcal{R}^{(j)}\triangleleft\mathcal{R}$ (a normal subgroup). Describe/identify the group $\mathcal{R}/\mathcal{R}^{(1)}$.
- (c) Similarly, define $\mathcal{K}^{(j)}$, prove that $\mathcal{K}^{(j)}\triangleleft\mathcal{K}$, and describe/identify $\mathcal{K}/\mathcal{K}^{(1)}$.
- (d) Prove: $x^3+y^4\overset{\mathcal{R}}{\sim}x^3+y^4+x^2y^2+xy^3$, $\quad x^p+y^p\overset{\mathcal{R}}{\sim}x^p+y^p+y^ax^{p-1}+x^by^{p-1}$, for $a,b\geq\frac{p-1}{2}$.
- (e) (Here $\Bbbk=\mathbb{C}$.) Let $f_t=x^p+y^q+z^r+txyz$, with $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$. Prove that $f_t\overset{\mathcal{K}}{\sim}f_1$ for any $t\neq 0$. Give a condition on t,t' to ensure $f_t\overset{\mathcal{R}}{\sim}f_{t'}$.
- (f) Disprove: i. $\phi(Jac(f))=Jac(\phi(f))$ for $\phi\in\mathcal{R}$. $\qquad$ ii. $\phi(Jac(f))=Jac(\phi(f))$ for $\phi\in\mathcal{K}$.
  iii. $\phi((f)+Jac(f))=\mathcal{K}\cdot\phi(f)+Jac(\mathcal{K}\cdot\phi(f))$. $\quad$ (What is the geometric interpretation?)
- (g) (Is there an equivalence weaker than $\mathcal{K}$?) Suppose for a certain equivalence $\sim$ on R holds: if $f\sim g$ then $V(f),V(g)\subset(\Bbbk^n,o)$ differ by a change of variables. Prove: if $f\sim g$ are square-free then $f\overset{\mathcal{K}}{\sim}g$.
- (h) Let $f=f_p+f_{>p}$ and $g=g_p+g_{>p}$. Prove: if $f\overset{\mathcal{R}}{\sim}g$ then $f_p\overset{GL_n(\Bbbk)}{\sim}g_p$, where n is the number of variables and the action $GL_n(\Bbbk)\circlearrowright\Bbbk^n$ is the usual one.
  Similarly prove: if $f\overset{\mathcal{K}}{\sim}g$ then $f_p\overset{GL_n(\Bbbk)}{\sim}(\pm)g_p$.
  ($V(f_p)\subset\Bbbk^n$ is called the tangent cone of $V(f)$, and $\mathcal{K}$-equivalence acts linearly on $V(f_p)$.)
- (i) Prove that the $\mathcal{K}$-orbits of the sets $\{x^4\}+\mathfrak{m}^5$, $\{x^3y\}+\mathfrak{m}^5$, $\{x^2y^2\}+\mathfrak{m}^5$ are disjoint.

(2) Below we assume: $f\in R=\Bbbk[[\underline{x}]]$ and the derivation $D=\sum\phi_i\frac{\partial}{\partial x_i}\in Der_{\Bbbk}(R)$ satisfies $D(\mathfrak{m})\subseteq\mathfrak{m}^2$.

- (a)
  - (i) Prove: $e^D(f)$ is a well defined power series, $e^D(f\cdot g)=e^D(f)\cdot e^D(g)$, and $e^D(f(\underline{x}))=f(e^D(\underline{x}))$.
  - (ii) Disprove: $e^D(f(\underline{x}))=f(\underline{x}+\underline{\phi})$
  - (iii) Suppose $e^{D_1}(f)=e^{D_2}(f)$ for any $f\in R$. Prove: $D_1=D_2$.
- (b) Let $\{D_j\}$ be a sequence of derivations satisfying: $D_j(\mathfrak{m})\subseteq\mathfrak{m}^{1+j}$. Prove: $\lim\limits_{j\to\infty}(e^{D_j}e^{D_{j-1}}\cdots e^{D_1})$ exists and is an automorphism of R.
- (c) Let $f\in\Bbbk\{x\}$, with the radius of convergence r. For which $a\in\Bbbk$ is $e^{a\frac{d}{dx}}(f)$ a well defined power series?

(3)

- (a) Find the order of $\mathcal{R}$-determinacy in the following cases
  i. $f(x,y)=x^3+y^k$, $\qquad$ ii. $f(x,y,z)=x^3+y^3+z^3$.
- (b) Let $f(x,y)=f_3(x,y)+f_{>3}(x,y)$. Suppose f_3 splits into 3 (pairwise) independent linear forms. Prove: $f\overset{\mathcal{R}}{\sim}xy(x-y)$.
- (c) We have proved the determinacy theorem for $\mathcal{R}$-equivalence. Finish the proof for $\mathcal{K}$-equivalence.
- (d) Define the filtration topology on R via the basic open sets: $\{f\}+\mathfrak{m}^j$, for $f\in R$, $j\in\mathbb{N}$.
  Verify: f is finitely $\mathcal{R}$ (resp. $\mathcal{K}$)-determined iff the orbit $\mathcal{R}\cdot f$ (resp. $\mathcal{K}\cdot f$) is open in this topology.

(4)

- (a) Verify: $dim\, R/_{Jac(f)}<\infty$ iff for some a holds: $dim\, R/_{\mathfrak{m}^a\cdot Jac(f)}<\infty$.
- (b) Verify: $dim\, R/_{Jac(f)+(f)}<\infty$ iff for some a holds: $dim\, R/_{\mathfrak{m}^a\cdot(Jac(f)+(f))}<\infty$.
- (c) Verify: if $\mu(f)<\infty$ then the critical locus of f is (set-theoretically) a point. Prove the converse statement for $R=\mathbb{C}\{\underline{x}\}$. Does the converse statement hold also for $R=\mathbb{R}\{\underline{x}\}$?