Geometric Calculus 2, 201.1.1041

The midterm, 19.05.2022, two hours.

(Lecturer: Dmitry Kerner)
No auxiliary material is allowed.
Solve all the questions. The total is 105 points.
Do not write in red color!

1. Fix the norm $\|A\| = \sqrt{trace(A \cdot A^t)}$ on $Mat_{2\times 2}(\mathbb{R})$, and identify $Mat_{2\times 2(\mathbb{R})} \cong \mathbb{R}^4$. Take the subset $\Sigma = \{A|\ det(A) = 0\} \subset Mat_{2\times 2}(\mathbb{R})$.

- **a.** (10 points) Prove: $\Sigma \setminus \{\mathbb{O}\}$ is a C^∞-submanifold of $Mat_{2\times 2(\mathbb{R})}$.
- **b.** (10 points) Specify charts and their transition maps for $\Sigma \setminus \mathbb{O}$.
- **c.** (10 points) Describe (explicitly) the tangent space $T_A\Sigma \subset Mat_{2\times 2}(\mathbb{R})$ for $A = \begin{bmatrix} \sqrt{3} & ln(2) \\ 0 & 0 \end{bmatrix}$.
- **d.** (15 points) Prove: the germ $(\Sigma, \mathbb{O})$ is not a C^1-submanifold of $Mat_{2\times 2}(\mathbb{R})$.

2. (10 points) Define the map $\mathbb{R}^2_{\underline{x}} \to \mathbb{R}^2_{\underline{y}}$ by $\underline{x} \to \underline{y} := A\underline{x}$, for an invertible matrix $A \in Mat_{2\times 2}(\mathbb{R})$. Find the vector field on $\mathbb{R}^2_{\underline{x}}$ whose pushforward is $\frac{\partial}{\partial y_1} - \frac{\partial}{\partial y_2}$.

3. (10 points) Define the map $(-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}^3$ by $t \to (sin(t), tan(t), e^t)$. Specify a 1-form on $\mathbb{R}^3$ whose pullback is $t^2 dt$.

4. (20 points) The support of a form $\omega \in \Omega^1(M)$ is defined as the closure, $supp(\omega) = \overline{\{p|\ \omega|_p \neq 0\}} \subseteq M$. For any covering by open sets $M = \cup \mathcal{U}_\alpha$ prove: $\omega = \sum \omega_\alpha$ where $supp(\omega_\alpha) \subseteq \mathcal{U}_\alpha$.

5. (20 points) Given a (differentiable) map $M \xrightarrow{\psi} \tilde{M}$, a function f on $\tilde{M}$, and a vector field ξ on M, prove: $\psi^*\big(\psi_*(\xi)(f)\big) = \xi(\psi^*(f))$.

Good Luck!