Geometric Calculus 2, 201.1.1041

Moed A, 05.07.2022, three hours.

(Lecturer: Dmitry Kerner)
No auxiliary material is allowed.
Solve all the questions. The total is 104 points.
Do not write in red color!

1. **a.** (7 points) Prove: if M is a path-connected manifold then the dimension of the germ $dim(M,p)$ does not depend on the point $p \in M$.

b. (11 points) Take a manifold with two charts, $M = \mathcal{U}_1 \cup \mathcal{U}_2$. Suppose the set $\mathcal{U}_1 \cap \mathcal{U}_2$ is path-connected. Prove: M is orientable.

2. **a.** (7 points) A form $\omega \in \overset{2}{\Lambda} V^*$ is called non-degenerate if for each $v \in V$ exists $u \in V$ such that $\omega(v,u) \neq 0$. Prove: if $dim(V)$ is odd then every 2-form is degenerate.

b. (11 points) A vector field ξ on $\mathbb{R}^N$ is called tangent to a submanifold $M \subset \mathbb{R}^N$ if $\xi|_p \in T_pM$ for each point $p \in M$. Suppose $M \subset \mathbb{R}^N$ is defined by the equation $f(\underline{x}) = 0$, with $grad(f)|_p \neq 0$ for each $p \in M$. Prove: ξ is tangent to M iff $\xi(f)|_M = 0$.

c. (7 points) Express the form $\omega = g(\|x\|) \cdot \sum x_i dx^i \in \Omega^1(\mathbb{R}^3 \setminus (0,0,0))$ in the polar coordinates. (Hint: no long computations.)

d. (11 points) Given a smooth map $\phi : M \to \tilde{M}$ and forms $\omega_1, \omega_2 \in \Omega^1(\tilde{M})$ prove: $\phi^*(\omega_1 \wedge \omega_2) = \phi^*(\omega_1) \wedge \phi^*(\omega_2)$.

3. (20 points) Compute $\int\limits_C \frac{-ydx+xdy}{x^2+y^2}$, where $C \subset \mathbb{R}^2$ is defined by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and is oriented counterclockwise.

4. **a.** (10 points) A subset $M \subset \mathbb{R}^{n+1}$ is defined by the equation $f(x) = 0$, where $f \in C^1(\mathbb{R}^{n+1})$ has no critical points on M, i.e. $f'|_p \neq 0$ for each point $p \in M$.
Prove: M is a smooth orientable hypersurface.

b. (20 points) Let $M \subset \mathbb{R}^{n+1}$ be a smooth orientable hypersurface. Prove: M can be defined by the equation $f(x) = 0$, where $f \in C^1(\mathcal{U})$ (for some neighborhood $M \subset \mathcal{U} \subset \mathbb{R}^{n+1}$) has no critical points on M.

Good Luck!