Geometric Calculus 2, 201.1.1041

Moed B, 24.07.2022, three hours.

(Lecturer: Dmitry Kerner)

No auxiliary material is allowed.

Solve all the questions. The total is 104 points.

Do not write in red color!

1. **a.** (10 points) Take an embedding of manifolds $M \overset{\psi}{\hookrightarrow} \tilde{M}$. Prove: the corresponding pusforward of vector fields, ψ_*, is injective.

b. (10 points) Take a finite dimensional vector space V. Take a skew-symmetric form $\omega \in \overset{k}{\Lambda} V^*$, where k is odd. Prove: $\omega \wedge \omega = 0$.

c. (10 points) Let M be an orientable path-connected manifold. Prove: M has exactly two (distinct) orientations.

2. **a.** (17 points) A subset $M \subset \mathbb{R}^{n+r}$ is defined by the equations $g_1(x) = 0 = \cdots = g_r(x)$ for some functions $g_1, \ldots, g_r \in C^1(\mathbb{R}^{n+r})$. Suppose the forms $dg_1, \ldots, dg_r \in \Omega^1(\mathbb{R}^{n+r})$ are linearly independent at each point of M. Take a function $f \in C^1(\mathbb{R}^{n+r})$. Prove: if the restriction $f|_M$ has a local extremum at a point $p \in M$ then $(df \wedge dg_1 \wedge \cdots \wedge dg_r)|_p = 0$.

b. (20 points) Fix a manifold M. Prove: for any discrete subset $X \subset M$ there exists a function $f \in C^1(M)$ satisfying: $f|_{M \setminus X} > 0$, $f|_X = 0$, and f has a non-degenerate minimum at each point $p \in X$. (Namely, in some local coordinates $f'|_p = 0$ and $f''|_p$ is non-degenerate.)

3. **a.** (17 points) Let M be an oriented compact manifold without boundary, $dim(M) = n$. Let $\omega \in \Omega^{n-1}(M)$ be continuously differentiable. Compute $\int_M d\omega$.

b. (20 points) Take the standard sphere $S^2 \subset \mathbb{R}^3$ with the orientation corresponding to the inner normal. Let $\omega = z dx \wedge dy|_{S^2} \in \Omega^2(S^2)$. Compute $\int_{S^2} \omega$.

Good Luck!