Geometric Calculus 2, 201.1.1041

Moed C, 4.09.2022, three hours.

(Lecturer: Dmitry Kerner)

No auxiliary material is allowed.

Solve all the questions. The total is 104 points.

Do not write in red color!

1. a. (10 points) Take a partition of unity $\sum \rho_\beta = 1$ on a manifold M, and a compact subset $X \subseteq M$. Prove: $X \cap supp(\rho_\beta) = \varnothing$ except for a finite set of β's.

b. (10 points) Prove: vectors $v_1, \dots, v_k \in V$ are linearly dependent iff $\omega(v_1, \dots, v_k) = 0$ for any form $\omega \in \overset{k}{\Lambda} V^*$.

c. (10 points) A curve $C \subset \mathbb{R}^2$ is defined via a (C^1) parametrization in polar coordinates, $r = r(\theta)$. Prove: $\int\limits_C f dC = \int\limits_{\theta_0}^{\theta_1} f \cdot \sqrt{(\partial_\theta r)^2 + r^2} d\theta$.

2. a. (18 points) Let $M \subset \mathbb{R}^n$ be a submanifold with boundary.
Prove: $\partial M \subset \mathbb{R}^n$ is a submanifold without boundary, i.e. $\partial\partial M = \varnothing$.

b. (18 points) Take a C^∞-manifold M and a closed subset $X \subset M$. Take a function $f \in C^\infty(M \setminus X)$ satisfying: $\lim\limits_{x \to x_o} f^{(k)}|_x = 0$ for each $x_o \in \partial X$ and all $k \geq 0$.
Prove: f extends to a C^∞-function on M. When is the extension unique?

3. a. (19 points) For an embedded manifold $M \subset \mathbb{R}^N$ and an integer $0 \leq k < \infty$ prove: the restriction map $\Omega^k(\mathbb{R}^N) \to \Omega^k(M)$ is surjective.

b. (19 points) Given an open bounded set $\mathcal{U} \subset \mathbb{R}^n$, suppose $\partial\overline{\mathcal{U}}$ is a compact manifold. Take the form $\omega = \sum c_i x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots dx_n \in \Omega^{n-1}(\mathbb{R}^n)$, where the constant $\{c_i\}$ satisfy $c_i^2 = 1$. Find $\{c_i\}$ that satisfy the identity $vol_n(\mathcal{U}) = \frac{1}{n} \int_{\partial\mathcal{U}} \omega$.

Good Luck!