Geometric Calculus 2, 201.1.1041

Homework 1

Spring 2022 (D.Kerner)

Questions to submit: 1. 2.b. 3.b. 3.f. 4.a. 6.b. 6.d.

1. Suppose $(\mathbb{R}^n, o) \xrightarrow{f} (\mathbb{R}^N, o)$ is C^1 and $rank[f'|_o] = k$. Prove: $rank[f'|_x] \geq k$ for x close to o. (Namely: for any representative $\mathcal{U} \xrightarrow{f} \mathbb{R}^N$ there exists an open neighborhood $o \in \tilde{\mathcal{U}} \subseteq \mathcal{U}$ such that ...)

2. Take a a C^r-function $(r \geq 1)$: $\mathbb{R}^n \supseteq \mathcal{U} \xrightarrow{f} \mathbb{R}^m$. Prove (using the implicit function theorem):
   a. (The normal form of a submersion, $m \leq n$) If $rank[f'|_p] = m$, then in some local (C^r) coordinates at $p \in \mathcal{U}$ (and the standard coordinates in $\mathbb{R}^m$) the function is: $f(\underline{x}) = f(p) + (x_1, \ldots, x_m)$.
   b. (The normal form of an immersion, $m \geq n$) If $rank[f'|_p] = n$, then in some local (C^r) coordinates at $f(p) \in \mathbb{R}^m$ (and the standard coordinates in $\mathbb{R}^n$) the function is: $f(\underline{x}) = (x_1 - f_1(p), \ldots, x_n - f_n(p), 0, \ldots, 0)$.
   c. Deduce the open mapping theorem: if $\mathbb{R}^n \supseteq \mathcal{U} \xrightarrow{f} \mathbb{R}^m$ is C^1, $m \leq n$, and $rank[f'] = m$ everywhere on $\mathcal{U}$ then f sends open sets to open sets.
   d. Let $\mathbb{R}^n \supseteq \mathcal{U} \xrightarrow{\phi} \tilde{\mathcal{U}} \subseteq \mathbb{R}^{\tilde{n}}$ be a C^1-diffeomorphism of open subsets. Prove: $n = \tilde{n}$.
   e. Suppose a subset $X \subset \mathbb{R}^n$ is defined by the equation $f(\underline{x}) = 0$. Suppose $\nabla(f)|_{x_o} \neq 0$. Prove: the tangent plane $T_{x_o}X \subset \mathbb{R}^n$ at a point $x_o \in X$ is defined by the equation $(\underline{x} - \underline{x}_o) \cdot \nabla(f)|_{x_o} = 0$.
   Extend this to the case of several equations.

3. a. We have defined the germs (X, x_o), f_{x_o} via an equivalence relation. Verify that this is indeed an equivalence relation.
   b. Prove: any (non-empty) open subgerm of $(\mathbb{R}^n, o)$ coincides with the germ $(Ball_\epsilon(o), o)$.
   c. Take the germ of a function $(\mathbb{R}^n, o) \xrightarrow{f_o} \mathbb{R}^1$. Write the full definition for the condition $f_o \geq 0$.
   d. Define the basic operations for a finite number of germs: $\cap_i(X_i, o) := (\cap_i X_i, o)$, $\cup_i(X_i, o) := (\cup_i X_i, o)$, $(X, o) \setminus (Y, o) := (X \setminus Y, o)$, $\prod_i (X_i, 0) := (\prod_i X_i, o)$. Verify: the results are well defined, i.e. do not depend on the choice of representatives. What happens in the infinite case?
   e. Define the basic operations on (a finite number of) function germs: $\sum_i (f_i)_p := (\sum_i f_i)_p$, $\prod_i (f_i)_p := (\prod_i f_i)_p$. Verify: the results are well defined.
   f. Let $0 \leq r \leq \infty$. Denote by $C^r(\mathbb{R}^p, o)$ the set of germs of C^r-functions. Verify: the derivatives at the origin, $f^{(j)}|_o$, $j = 0, \ldots, r$, are well defined, i.e. do not depend on the choice of a representative of f. The set $C^r(\mathbb{R}^p, o)$ is a commutative, unital ring.

4. Prove: the following sets are C^∞-manifolds. Identify these manifolds, e.g. $X_2 \cong S^1 \times S^1$ (a C^∞-diffeomorphism).
   a. $X_n = \{(x, y) | \ \|x\| = \|y\| = 1\} \subset \mathbb{R}^n \times \mathbb{R}^n$.
   b. $\{(x, y, z) | \ x^2 + y^2 = 0, \ x^2 + y^2 + z^2 = 2x\} \subset \mathbb{R}^3$.
   c. $\{(x_1, \ldots, x_4) | \ \|x\| = 1, \ x_1 x_2 + x_3 x_4 = 0\} \subset \mathbb{R}^4$.

5. Fix a(ny) norm on $Mat_{n \times n}(\mathbb{R})$, and identify $Mat_{n \times n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$. Prove: the subsets $SO(n, \mathbb{R}), SL(n, \mathbb{R}) \subset O(n, \mathbb{R}) \subset GL(n, \mathbb{R}) \subset Mat_{n \times n}(\mathbb{R})$ are C^∞-manifolds. Determine their dimensions. Which of these are compact/path-connected?

6. a. Prove: $(C, o) = \{(x, y) | \ y = |x|\} \subset (\mathbb{R}^2, o)$ is not the germ of a C^1-manifold.
   Prove: $\mathbb{R}^1 \ni t \to (t^7, |t|^7) \in \mathbb{R}^2$ is a C^6-parametrization of C. Any contradiction?
   b. Prove: the germ $(C, o) = \{(x, y) | \ x \cdot y = 0\} \subset (\mathbb{R}^2, o)$ is not the germ of a C^0-manifold. (i.e. (C, o) is not homeomorphic to $(\mathbb{R}^n, o)$ for any n.)
   c. Define the curve $C \subset \mathbb{R}^2$ by the parametrization $\mathbb{R}^1 \ni t \to (t^3, t^5) \in \mathbb{R}^2$. Prove: C is a C^1 manifold, but not a C^2-manifold. Give an (explicit) non-degenerate C^1-parametrization of C.
   d. For each $r \geq 1$ give an example of C^r-manifold that is not a C^{r+1}-manifold.
   e. Prove: the dimension of a path-connected manifold is well defined. (i.e. $dim_p M$ does not depend on p)