Geometric Calculus 2, 201.1.1041

Homework 10

Spring 2022 (D.Kerner)

Questions to submit: 1.c. 1.d. 1.f. 1.g. 3.b. 4.a. 4.b. 4.c. 4.f. 4.g.

Below $\mathcal{U} \subseteq \mathbb{R}^n$ is an open subset with the standard coordinates of $\mathbb{R}^n$.
∂M always denotes the manifold-boundary (not the boundary in the sense of the topology on $\mathbb{R}^N$)

1. a. In the class we have associated to a vector field $\vec{F}$ on M the form $\omega^{n-1}_{\vec{F}} \in \Omega^{n-1}(M)$. For $M = \mathcal{U}$ and $\vec{F} = \sum F_i \frac{\partial}{\partial x^i}$ on $\mathcal{U}$ write this form explicitly.
   b. Associate to a vector field $\vec{F} = \sum F_i \frac{\partial}{\partial x_i}$ the 1-form $\omega^1_{\vec{F}} := \sum F_i \cdot dx^i$. Verify: $\omega^1_{grad(f)} = df$.
   c. Associate to a scalar function $f \in C^r(\mathcal{U})$ the n-form $\omega^n_f := f \cdot dx^1 \wedge \cdots \wedge dx^n$. For $\mathcal{U} \subseteq \mathbb{R}^3$ verify:
   i. $\omega^2_{\vec{F}\times\vec{G}} = \omega^1_{\vec{F}} \wedge \omega^1_{\vec{G}}$ ii. $\omega^2_{\vec{F}} \wedge \omega^1_{\vec{G}} = \omega^3_{\vec{F}\cdot\vec{G}}$ iii. $\omega^2_{rot(\vec{F})} = d\omega^1_{\vec{F}}$ iv. $\omega^3_{div(\vec{F})} = d\omega^2_{\vec{F}}$.
   d. Verify: the flux of $\vec{F}$ through a smooth oriented surface $M \subset \mathbb{R}^3$ equals to $\int_{\vec{S}} \omega^2_{\vec{F}}$.
   e. Take a smooth hypersurface-germ $(M, p) \subset (\mathbb{R}^{n+1}, p)$. Fix some linearly independent vectors $v_1 \dots v_n \in T_pM$. Define the form $\omega_{v_1\dots v_n} \in T^*_p\mathbb{R}^{n+1}$ by $\omega_{v_1\dots v_n}(v) = det[v, v_1, \dots, v_n]$. Prove: $\omega_{v_1\dots v_n}$, when presented in the canonical basis of $\mathbb{R}^{n+1}$, gives the normal $\vec{n}_p$ to M at p.
   f. Prove: for a smooth hypersurface $M \subset \mathbb{R}^{n+1}$ (with no prescribed orientation) choosing the volume form is equivalent to choosing the unit normal.
   g. Let $M \subset \mathbb{R}^{n+1}$ be a smooth orientable hypersurface. Prove: M can be defined as $\{f = 0\}$, where f has no critical points on M.

2. We have defined $\int_M \omega$ via the splitting $M \setminus X_{\leq n-1} = \coprod M_i$. Given some charts on a manifold, $M = \cup \mathcal{U}_\alpha$, and a corresponding partition of unity, $\sum \rho_\alpha = 1$, verify: $\int_M \omega = \sum_\alpha \int_{\mathcal{U}_{\alpha,x}} \phi^*_\alpha(\rho_\alpha \omega)$.
   In particular, the right hand side is independent of the choice of charts and the choice of the partition.

3. We have defined (non-embedded) manifolds with boundary. A subset $M \subset \mathbb{R}^N$ is called a (C^r) *sub*manifold with boundary if the germ $(\mathbb{R}^N, M, p)$ is C^r-diffeomorphic to the germ $(\mathbb{R}^N, H, q)$ for each point $p \in M$ and a corresponding point $q \in H$.
   a. Prove: in this case ∂M is a submanifold of $\mathbb{R}^N$, with $\partial(\partial M) = \varnothing$.
   In particular, solid boxes/polytopes in $\mathbb{R}^{n>1}$ are not submanifolds with boundary.
   b. Take $\mathcal{U} \subset \mathbb{R}^n$ with the standard orientation. Which orientation is induced on $\partial\bar{\mathcal{U}}$? (Is the normal to $\partial\bar{\mathcal{U}}$ outer or inner?)
   c. Take $S^n_+ := S^n \cap \{x_1 \geq 0\}$. What is the induced orientation on ∂S^n_+?
   d. Find (many) mistakes (give counterexamples) in the statement: for an open $\mathcal{U} \subset \mathbb{R}^n$ the boundary is a manifold of dimension $(n-1)$, with $\partial(\partial\mathcal{U}) = \varnothing$, and with the orientation naturally induced from $\mathcal{U}$.

4. a. Let $\mathcal{U} \subset \mathbb{R}^3$ be an open convex bounded set. Suppose $(\partial\bar{\mathcal{U}}) \setminus X_{\leq 1}$ is smooth. Take $\omega = f_x dy \wedge dz + f_y dz \wedge dx + f_z dx \wedge dy|_{\partial\bar{\mathcal{U}}}$, where f_i is independent of x_i. Prove: $\int_{\partial\bar{\mathcal{U}}} \omega = 0$.
   b. Compute $\int_S \omega$ for $S \subset \mathbb{R}^n$ defined by $\sum \frac{x_i^2}{a_i^2} = 1$ (with the outer normal) and $\omega = \frac{\sum(-1)^i x_i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n}{\|\underline{x}\|^n}$.
   c. For a smooth surface with boundary $S \subset \mathbb{R}^N$ and $\omega = g(\|x\|) \cdot \sum x_i dx^i|_S \in \Omega^1(S)$ with $g \in C^1$, prove: $\int_{\partial S} \omega = 0$.
   d. (Integration by parts in a multiple integral) Suppose $\bar{\mathcal{U}} \subset \mathbb{R}^n$ is compact, and its boundary $S := \partial\bar{\mathcal{U}}$ is piecewise smooth, with the outer unit normal $\hat{n}$. Prove: $\int_{\bar{\mathcal{U}}} (\partial_i f) \cdot g \cdot d\bar{\mathcal{U}} = \int_S f \cdot g \cdot \hat{n}_i \cdot d\vec{S} - \int_{\bar{\mathcal{U}}} f \cdot (\partial_i g) \cdot d\bar{\mathcal{U}}$.
   e. For a body $\bar{\mathcal{U}} \subset \mathbb{R}^3$ obtain the formula $vol_3(\bar{\mathcal{U}}) = \frac{1}{3} \int_{\partial\bar{\mathcal{U}}} x dy \wedge dz + y dz \wedge dx + z dx \wedge dy$.
   f. Find the smooth, closed oriented curve C for which the integral $\oint_C (\frac{x^2 y}{4} + \frac{y^3}{3}) dx + x dy$ achieves maximum.
   g. Suppose $\bar{\mathcal{U}} \subset \mathbb{R}^2$ is compact and $\partial\bar{\mathcal{U}}$ is piecewise smooth and positively oriented. Prove: $vol_2(\bar{\mathcal{U}}) = \oint\limits_{\partial\bar{\mathcal{U}}} x dy = -\oint\limits_{\partial\bar{\mathcal{U}}} y dx$. Use this formula to compute the area bounded by the curve $|\frac{x}{a}|^{\frac{2}{p}} + |\frac{y}{b}|^{\frac{2}{p}} = 1$.