Geometric Calculus 2, 201.1.1041

Homework 2

Spring 2022 (D.Kerner)

Questions to submit: 1. 2.a. 2.b. 2.d. 3.b. 5.b. 6.c.

1. For a manifold-germ $(M,o) \subset (\mathbb{R}^N, o)$ and a parameterized curve germ $(\mathbb{R}^1, o) \ni t \to x(t) \in (M,o)$ take the velocity $\frac{dx(t)}{dt}|_{t=o} \in \mathbb{R}^N$. Prove: T_oM is the union of such velocity vectors over all the curve germs.

2. Define the standard sphere $S^n \subset \mathbb{R}^{n+1}$ by the equation $\|\underline{x}\| = 1$. Define the projections:

$$S^n \setminus \{\hat{x}_{n+1}\} =: \mathcal{U}_- \ni \underline{x} \xrightarrow{\phi_-} \frac{(x_1,\dots,x_n,0)}{1-x_{n+1}} \in \mathbb{R}^n_-, \qquad S^n \setminus \{-\hat{x}_{n+1}\} =: \mathcal{U}_+ \ni \underline{x} \xrightarrow{\phi_+} \frac{(x_1,\dots,x_n,0)}{1+x_{n+1}} \in \mathbb{R}^n_+.$$

   a. Determine the geometric meaning of this projections. In particular: i. Verify the injectivity. ii. What are the limits $\lim\limits_{x\to\hat{x}_{n+1}} \phi_-(x)$, $\lim\limits_{x\to-\hat{x}_{n+1}} \phi_+(x)$? iii. Describe the closures $\overline{\phi_-(\mathcal{U}_-)} \subseteq \mathbb{R}^n_-$, $\overline{\phi_+(\mathcal{U}_+)} \subseteq \mathbb{R}^n_-$.
   b. Verify: the map $\mathbb{R}^n_+ \setminus \{o\} \overset{\phi_- \circ (\phi_+)^{-1}}{\to} \mathbb{R}^n_- \setminus \{o\}$ is given by $\underline{y} \to \frac{\underline{y}}{\|\underline{y}\|^2}$.
   c. Show that the charts $(\mathcal{U}_-, \phi_-)$, $(\mathcal{U}_+, \phi_+)$ form a C^∞-atlas on S^n.
   d. Prove: every C^0-atlas on S^n must have at least two charts.

3. a. A torus in $\mathbb{R}^3$ with radii $0 < r < R < \infty$ is obtained by rotating the circle $(x-R)^2 + z^2 = r^2$ around the $\hat{z}$-axis. Write the defining equation of the torus. Write the parametrization of the torus. What is the minimal number of C^1-charts for this torus?
   b. Let M be a compact C^1-manifold of dimension n. Prove: M is non-embeddable into $\mathbb{R}^n$.

4. Take all the balls in $\mathbb{R}^n$ centered at the points of $\mathbb{Q}^n \subset \mathbb{R}^n$, and with radii belonging to the sequence $\{\frac{1}{p_k}\}_{k\in\mathbb{N}}$, where p_k is the k'th prime number. Prove: this is a base for the standard topology on $\mathbb{R}^n$.

5. Fix a topological space $(X, \mathcal{T}_X)$.
   a. Prove: i. $\varnothing, X$ are closed subsets. ii. Any intersection of closed subsets of X is a closed subset. iii. Any finite union of closed subsets of X is a closed subset.
   b. For a subset $Y \subset X$ take the induced topology, $\mathcal{T}_Y = \mathcal{T}_X|_Y$. Take a subset $Z \subset Y \subset X$. (Dis)prove: i. Z is closed in Y iff Z is closed in X. ii. Z is open in Y iff Z is open in X. iii. (for the closures) $\overline{Z}^{(Y)} = \overline{Z}^{(X)}$. iv. (for the interiors) $Int_Y(Z) = Int_X(Z) \cap Y$.

6. Fix a topological space X, Hausdorff and with a countable base. Consider various C^r-atlases on X.
   a. Prove: every atlas is contained in a (unique) maximal (w.r.t. inclusion) atlas.
   b. Prove: two atlases on X are compatible iff they have the same maximal atlas.
   c. Take an atlas $\{\mathcal{U}_\alpha, \phi_\alpha\}$ on X and two other charts, $(\mathcal{V}_1, \psi_1)$, $(\mathcal{V}_2, \psi_2)$, not from this atlas. Suppose for each point $x \in \mathcal{V}_1 \cap \mathcal{V}_2$ exists a chart $x \in \mathcal{U}_\alpha \subseteq X$ that is compatible with $\mathcal{V}_1$ and $\mathcal{V}_2$. Prove: the charts $\mathcal{V}_1, \mathcal{V}_2$ are compatible.
   d. Suppose X admits a C^0-atlas $\mathcal{A}$ with just one chart. Prove: $(X, \mathcal{A})$ is a C^∞-manifold which is C^∞-diffeomorphic to an open subset of $\mathbb{R}^n$.