Geometric Calculus 2, 201.1.1041
Homework 3

Spring 2022 (D.Kerner)

Questions to submit: 1.a. 1.b. 1.c. 2.b. 2.c. 3.c.ii. 3.f.ii. 4.b. 4.d.

Below M is a C^r-manifold, $1 \le r \le \infty$, $dim(M) = n$, with coordinate charts $M = \cup(\mathcal{U}_\alpha, \phi_\alpha)$.

1. a. Consider the subset of $\mathbb{R}^3$ defined by the equations $x^3 + y^3 + z^3 = 1$, $2x^6 + 2y^6 + z^6 = 1$. Prove: except for a finite number of points this is a smooth (possibly empty) curve. Find these "special" points.
   b. Suppose M is path-connected, $dim(M) = 1$. Prove: either $M \cong \mathbb{R}^1$ or $M \cong S^1$.
   c. The curve $C \subset \mathbb{R}^2$ is defined by the equation $y^2 = x^3$. Give a C^∞-atlas on C that is not a sub-atlas of the C^1-atlas induced from $\mathbb{R}^2$. (Thus C is a C^∞-manifold, but not a C^1-submanifold of $\mathbb{R}^2$.)

2. For all the questions below verify that your considerations do not depend on the choice of a chart.
   a. Define the notion of convergent sequence of points on M.
   b. Is the notion "Cauchy sequence of points" well defined on M?
   c. Given a C^2-function $f : M \to \mathbb{R}^1$ define the notions of critical point, local min/max. State/prove the criterion for the local min/max in terms of $f^{(2)}$. (Namely, reduce the proof to the case of Calculus 3.)

3. a. (Dis)Prove: a subset $\mathcal{U} \subset M$ is open/closed iff $\phi_\alpha(\mathcal{U} \cap \mathcal{U}_\alpha)$ is open/closed in $\mathbb{R}^n$ for each α.
   b. Prove: one can assume that all the opens $\phi_\alpha(\mathcal{U}_\alpha) \subset \mathbb{R}^n$ are bounded. Namely, compose each ϕ_α with a C^∞-diffeomorphism $\psi \circlearrowright \mathbb{R}^n$ (construct it) such that ....
   c. Take a C^1-map $M \xrightarrow{\phi} \tilde{M}$ with $dim(M) = dim(\tilde{M})$ and $\phi'|_p$ of the full rank for each $p \in M$.
      i. Prove: ϕ is an open map, i.e. $\phi(\mathcal{U}) \subseteq \tilde{M}$ is open for each open $\mathcal{U} \subseteq M$.
      ii. In the particular case $\mathbb{R}^n \xrightarrow{\phi} \mathbb{R}^n$ is ϕ injective?
   d. Prove: a C^r-map $M \xrightarrow{\phi} \tilde{M}$ induces the homomorphism of (commutative, unital) rings $C^r(\tilde{M}) \xrightarrow{\phi^*} C^r(M)$. (Dis)Prove: $f \in C^r(\tilde{M})$ iff $\phi^*(f) \in C^r(M)$.
   e. Suppose a set-theoretic map $\phi : M \to \tilde{M}$ satisfies: $\phi^*(f) \in C^r(M)$ for each $f \in C^r(\tilde{M})$. (Dis)Prove: ϕ is a C^r-map of manifolds.
   f. Suppose a C^r-map $\phi : M \to \tilde{M}$ $(r \ge 1)$ is a bijection onto its image, and the derivative ϕ' is of full rank everywhere on M.
      i. Does this imply that ϕ is an embedding? We have seen the example of number 6 in $\mathbb{R}^2$. Construct also another example: an injective analytic map $\mathbb{R}^1 \to S^1 \times S^1$ with the dense image. (You can use the fact: the sequence $\{n \cdot c \ mod \ \mathbb{Z}\}$ is dense in $[0, 1]$ provided $c \notin \mathbb{Q}$.)
      ii. Prove: if in addition M is compact, then ϕ is an embedding.

4. a. Give an example of M with an open subset that is not a coordinate chart for any atlas. Prove (for a maximal atlas): any open $\mathcal{U} \subseteq M$ is the union of some coordinate charts, $\mathcal{U} = \cup \mathcal{U}_\alpha$.
   b. Take a partition of unity $\sum \rho_\beta = 1$ and a compact subset $X \subseteq M$. Prove: $X \cap supp(\rho_\beta) = \varnothing$ except for a finite set of β's.
   c. Take a maximal atlas on M and an open covering $M = \cup \mathcal{U}_\alpha$. Prove that there exists a partition of unity, $\sum \rho_\beta = 1$, satisfying: $supp(\rho_\beta) \subseteq \mathcal{U}_{\alpha(\beta)}$ (for each β and a corresponding $\alpha(\beta)$) and the sets $\{supp(\rho_\beta)\}$ are compact.
   d. Prove: for any open subset $\mathcal{U} \subset M$ there exists a "bump" function $\rho \in C^r(M)$ satisfying: $\rho|_{\mathcal{U}} > 0$, $\rho|_{M \setminus \mathcal{U}} = 0$.