Geometric Calculus 2, 201.1.1041

Homework 4

Spring 2022 (D.Kerner)

Questions to submit: 2.a. 2.b. 2.d.ii. 2.e. 3.d. 4.c. 4.d. 4.e.

Below M is a C^r-manifold, $1 \le r \le \infty$, $dim(M) = n$, with coordinate charts $M = \cup(\mathcal{U}_\alpha, \phi_\alpha)$.

1. Take a C^∞-manifold M and a closed subset $X \subset M$. Take a function $f \in C^\infty(M \setminus X)$ satisfying: $\lim\limits_{x \to x_o} f^{(k)}(x)=0$ for each $x_o \in \partial X$ and all $k \ge 0$. Prove: f extends to $C^\infty(M)$. When is the extension unique?

2. A group G is called "a Lie group" if it is a C^1-manifold and the group operations (the product and the inverse) are C^1-maps.
   a. You have proved in homework.1 that the groups $GL(n,\mathbb{R})$, $SL(n,\mathbb{R})$, $O(n,\mathbb{R})$, $SO(n,\mathbb{R})$ are C^∞-submanifolds of $Mat_{n\times n}(\mathbb{R})$. Verify: these are Lie groups.
   b. Compute the total derivative for the following maps:
   i. The inverse map $GL(n,\mathbb{R}) \ni A \stackrel{\phi}{\to} A^{-1} \in GL(n,\mathbb{R})$. ii. The k-th power map $Mat_{n\times n}(\mathbb{R}) \ni A \stackrel{\phi}{\to} A^k \in Mat_{n\times n}(\mathbb{R})$. (Express the answer as a linear form, do not try to write down the partial derivatives.)
   c. Establish the C^∞-diffeomorphisms:
   i. $SO(2) \cong S^1$, $O(2) \cong S^1 \coprod S^1$. ii. $SL(2,\mathbb{R}) \cong \{(x,y,z,w) |\ x^2+y^2 = 1+z^2+w^2\} \subset \mathbb{R}^4$.
   d. Prove the following identifications of the tangent spaces at the unit matrix, $T_{\mathbf{I}}G \subseteq Mat_{n\times n}(\mathbb{R})$:
   i. $\mathfrak{gl}(n,\mathbb{R}) := T_{\mathbf{I}}GL(n,\mathbb{R}) = Mat_{n\times n}(\mathbb{R})$ ii. $\mathfrak{sl}(n,\mathbb{R}) := T_{\mathbf{I}}SL(n,\mathbb{R}) =$(matrices with zero trace)
   iii. $\mathfrak{so}(n,\mathbb{R}) := T_{\mathbf{I}}SO(n,\mathbb{R}) =$(skew-symmetric matrices) iv. $o(n,\mathbb{R}) := T_{\mathbf{I}}O(n,\mathbb{R}) = T_{\mathbf{I}}SO(n,\mathbb{R})$.
   e. Compute the action of the linear form ϕ' from part b. on these tangent spaces.
   f. A vector subspace $V \subseteq Mat_{n\times n}(\mathbb{R})$ is called "a Lie algebra" if it is closed under the commutator, i.e. $[A,B] := AB - BA \in V$ for all $A, B \in V$. Prove: $\mathfrak{gl}(n,\mathbb{R})$, $\mathfrak{sl}(n,\mathbb{R})$, $\mathfrak{so}(n,\mathbb{R})$ are Lie algebras.

3. Introduce equivalence relation on $\mathbb{R}^{n+1} \setminus \{o\}$ by: $v \sim w$ if $v \in \mathbb{R}\cdot w$. The real projective space $\mathbb{RP}^n$ is the set of equivalence classes. Take the natural projection $\psi : \mathbb{R}^{n+1} \setminus \{o\} \to \mathbb{RP}^n$. The points of $\mathbb{RP}^n$ are denoted by $(x_0 : \dots : x_n)$ (defined up to scaling). Here $x_0, \dots, x_n$ are called "the homogeneous coordinates".
   a. As the open sets on $\mathbb{RP}^n$ one takes the ψ-images of the opens in $\mathbb{R}^{n+1} \setminus \{o\}$. Verify: this is a topology.
   b. Take the charts $\mathcal{U}_j = \{(x_0 : \dots : x_n) |\ x_j \neq 0\} \subset \mathbb{RP}^n$ with the coordinate maps
   $$\mathcal{U}_j \ni (x_0 : \dots : x_n) \stackrel{\phi_j}{\to} (\frac{x_0}{x_j}, \dots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \dots \frac{x_n}{x_j}) \in \mathbb{R}^n.$$
   Verify: these maps are well defined homeomorphisms. Write down the transition functions. Conclude: $\mathbb{RP}^n$ is a C^∞-manifold.
   c. Prove: $\psi : \mathbb{R}^{n+1} \setminus \{o\} \to \mathbb{RP}^n$ is a C^∞-map of manifolds, and it factorizes as $\mathbb{R}^{n+1} \setminus \{o\} \to S^n \stackrel{2:1}{\to} \mathbb{RP}^n$.
   d. Prove: $\mathbb{RP}^1 \cong S^1$.
   e. Can you construct $\mathbb{RP}^2$ topologically by gluing the $A4$-page? (As with Möbius strip and Klein bottle).

4. a. Given a vector field ξ on M and a function $f \in C^1(M)$ verify: $\xi(f)$ is a well defined function.
   b. Prove the Leibnitz rule: $\xi(f \cdot g) = \cdots$.
   c. Which of the following vector fields on $\mathbb{R}^1$ are related by coordinate changes:
   $2sin(x)\frac{d}{dx}$, $sin^2(x)\frac{d}{dx}$, $sin(2x)\frac{d}{dx}$.
   d. Express the vector fields $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ on $\mathbb{R}^2$ in the polar coordinates. Express the vector fields $\frac{\partial}{\partial r}$, $\frac{\partial}{\partial \phi}$ in the cartesian coordinates. Compute the commutator $[\frac{\partial}{\partial r}, \frac{\partial}{\partial \phi}]$.
   e. Take the "stereographic" covering $S^1 = \mathbb{R}^1_+ \cup \mathbb{R}^1_-$, with coordinates x, y. Write the pushforward (transition) map for vector fields from $\mathbb{R}^1_+$ to $\mathbb{R}^1_-$. Prove: if a vector field is polynomial in both charts then the degree of this polynomial is ≤ 2.
   f. Prove: any C^r-vector field on S^1 is presentable as $c(\phi)\frac{\partial}{\partial \phi}$ for some $c \in C^r(S^1)$.