Geometric Calculus 2, 201.1.1041

Homework 5

Spring 2022 (D.Kerner)

Questions to submit: 1.a. 2.c. 2.d. 2.f.iii. 3.d. 4.a. 4.b. 4.c. 5.d.ii.

1. a. Two map-germs $\gamma_1, \gamma_2 : (\mathbb{R}^1, o) \to (M, p)$ are called equivalent if $\frac{d\phi_\alpha \circ \gamma_1}{dt}|_{t=0} = \frac{d\phi_\alpha \circ \gamma_2}{dt}|_{t=0}$ for some chart. Prove: γ_1, γ_2 are equivalent iff the corresponding paths are tangent. Namely, $\|\phi_\alpha \circ \gamma_1(t) - \phi_\alpha \circ \gamma_2(t)\| = o(t)$.
   b. Prove: if $M \subset \mathbb{R}^N$ then every vector $v \in T_pM$ has the unique representative $\tilde{v} \in \mathbb{R}^N$, which is tangent to M at p.

2. Given a vector field ξ and $f \in C^r(M)$ define the function $M \stackrel{\xi(f)}{\to} \mathbb{R}^1$ pointwise, $\xi(f)|_p := \frac{df(\gamma(t))}{dt}|_{t=0}$, where $\gamma : (\mathbb{R}^1, o) \to (M, p)$ is a map that represents $\xi|_p \in T_pM$.
   a. Prove: the function $\xi(f)$ is well defined.
   b. Suppose in some chart ξ is presented as $\sum c_i(x)\frac{\partial}{\partial x_i}$. Verify (in that chart): $\xi(f) = \sum c_i(x)\partial_{x_i} f(x)$.
   c. Assuming ξ is a C^{r-1}-field, prove: the application $(\xi, f) \to \xi(f)$ defines the $\mathbb{R}$-linear map $\xi : C^r(M) \to C^{r-1}(M)$. Establish the Leibniz rule, $\xi(f \cdot g) = \cdots$
   Prove: ξ is uniquely determined by this map, i.e. if $\xi(f) = \tilde{\xi}(f)$ for all $f \in C^r(M)$ then $\xi = \tilde{\xi}$.
   d. Prove (for a C^2-manifold M): the commutator of two vector fields, i.e. $[\xi, \eta] := \xi \circ \eta - \eta \circ \xi$, is a vector field. Namely, this is a differential operator of order one, and its transformations under transition maps are those of a vector field. One says: the vector space of vector fields on M is a Lie algebra.
   e. For vector fields ξ, η on M verify: $[f\xi, g\eta] = fg[\xi, \eta] - g\eta(f)\xi + f\xi(g)\eta$ for any $f, g \in C^1(M)$.
   f. A vector field ξ on $\tilde{M}$ is called tangent to a submanifold $M \subset \tilde{M}$ if $\xi|_p \in T_pM$ for each point $p \in M$.
      i. Verify: the restriction $\xi|_M$ is a vector field on M iff ξ is tangent to M.
      ii. Prove: if ξ, η are tangent to M then $\xi + \eta$ and $[\xi, \eta]$ are tangent as well.
      iii. Suppose $M \subset \mathbb{R}^N$ is defined by $f(\underline{x}) = 0$, with $grad(f)|_p \neq 0$ for each $p \in M$. Prove: ξ is tangent to M iff $\xi(f)|_M = 0$.

3. a. Take a function $f \in C^r(\mathcal{U})$ for an open $\mathcal{U} \subseteq \mathbb{R}^n$. Define $\omega = df$ and $\xi = grad(f)$. Compute $\omega(\xi)$.
   b. Verify: the pairing of forms and vector fields, $(\omega, \xi) \to \omega(\xi)$, is $C^r(M)$-bilinear, i.e.
   $\omega(\xi_1 + \xi_2) = \ldots,$ $\omega(f \cdot \xi) = f \cdot (\omega(\xi)) = (f \cdot \omega)(\xi),$ $(\omega_1 + \omega_2)\xi = \ldots.$
   c. We have defined the map $C^r(M) \to \Omega^1(M)$, $f \to df$. Verify: it is $\mathbb{R}$-linear, and satisfies the Leibniz rule.
   d. A submanifold $M \subset \mathbb{R}^{n+1}$ is defined by the equation $f(\underline{x}) = 0$ for some $f \in C^r(\mathbb{R}^N)$. Compute $df|_M$.

4. Take the polar coordinates (r, θ, ϕ) on $\mathbb{R}^3$, with $\phi \in [0, 2\pi]$ and $\theta \in [0, \pi]$.
   a. Take the charts $S^2 = \mathbb{R}^2_+ \cup \mathbb{R}^2_-$. Take the vector fields $\partial_{x_1}, \partial_{x_2}$ on $\mathbb{R}^2_+$. Write their pushforwards on $\mathbb{R}^2_-$. Write these vector fields in the polar coordinates on S^2.
   b. Do the vector fields $sin(\theta)\frac{\partial}{\partial \theta}$, $\frac{\partial}{\partial \phi}$ extend to the global C^∞-fields on S^2? Compute the commutator $[sin(\theta)\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi}]$. Can you explain the result geometrically?
   c. Express the forms $dx, dy, dz \in \Omega^1(\mathbb{R}^3)$ in polar coordinates. Express the 1-forms dr, $d\theta$, $d\phi$ in cartesian coordinates.

5. a. Restrict the form $\omega = xdx + ydy$ onto $S^1 \subset \mathbb{R}^2$. Present the restriction in the polar coordinates on S^1.
   b. Take the torus $S^1 \times S^1 \subset \mathbb{R}^3$ from question 2.a of homework 2. Take $\omega = \sum \omega_i dx_i \in \Omega^1(\mathbb{R}^3)$. Present the restriction $\omega|_{S^1 \times S^1}$ in the coordinates (ϕ_1, ϕ_2) on this torus.
   c. Take a C^r map $\psi : (M, p) \to (\tilde{M}, \tilde{p})$. (Dis)Prove: i. If ψ is an embedding then ψ_* is injective. ii. If ψ is surjective then ψ_* is surjective. iii. $\psi^*((\psi_*\xi)(f)) = \xi(\psi^*(f))$ for each $f \in C^r(\tilde{M})$.
   d. Given a map $\psi : M \to \tilde{M}$ and the corresponding homomorphism of vector spaces $\psi^* : \Omega^1(M) \leftarrow \Omega^1(\tilde{M})$.
      i. Verify: $\psi^*(f \cdot \omega) = \psi^*(f) \cdot \psi^*(\omega)$.
      ii. Verify: $\psi^* df = d\psi^* f$. For a submanifold $M \subseteq \tilde{M}$ one gets: $(df)|_M = d(f|_M)$.
      iii. Given two maps $M_1 \stackrel{\psi_1}{\to} M_2 \stackrel{\psi_2}{\to} M_3$ prove: $(\psi_2 \circ \psi_1)^* = \psi_1^* \circ \psi_2^*$.