Geometric Calculus 2, 201.1.1041

Homework 6

Spring 2022 (D.Kerner)

Questions to submit: 1. 2.b. 2.d. 3.a. 3.e. 4.b.iii. 4.c.ii. 4.d. 4.f.

1. For a point $p \in M$ denote by $\mathfrak{m}_p \subset C^\infty(M)$ the subset of functions that vanish at p. Prove: $\mathfrak{m}_p$ is an ideal in the ring $C^\infty(M)$. Define the map $C^\infty(M) \to T_p^*M$ by $f \to df|_p$. Prove: its kernel is $\mathfrak{m}_p^2$.

2. a. We have defined the (scalar) integral $\int\limits_C f dC$ via partitions on C, and for these we have (slightly) used the parametrization of C. Verify: the integral (if exists) does not depend on the choice of parametrization.
   b. Compute the integrals: i. $\int\limits_{\{x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}\}} (x^{\frac{4}{3}} + y^{\frac{4}{3}})dC$. ii. $\int\limits_{(x^2+y^2)^2=a^2(x^2-y^2)} |y|dC$.
   c. For which values of $s > 0$ the planar curve defined by $r(\theta) = \frac{1}{1+\theta^s}$, $\theta \in [0, \infty)$ has a finite length?
   d. A curve $C \subset \mathbb{R}^3$ is defined via parameterization in polar coordinates by equations $r = r(\phi)$, $\theta = \theta(\phi)$, for $\phi \in [\phi_0, \phi_1]$. (Here θ is the angle with $\hat{z}$-axis.) Prove: $\int_C f \cdot dC = \int_{\phi_0}^{\phi_1} f \cdot \sqrt{(\partial_\phi r)^2 + r^2(\partial_\phi \theta)^2 + r^2 sin^2(\theta)} d\phi$.
   e. Suppose $\int_C f dC$ exists. Prove: $|\int_C f dC| \le \int_C |f| dC$.

3. a. Compute $\int_C \vec{F} \cdot d\vec{C}$ in the following cases:
   i. $\vec{F}(\underline{x}) = (x_1, x_2^2, \dots, x_n^n)$, $C = \{(sin(t), sin^2(t), \dots, sin^n(t)) |\ t \in [0, \pi]\}$.
   ii. $\int\limits_{\left\{\substack{x^2+y^2+z^2=1, \\ y=-x, \quad z>0}\right\}} (z^2 dx + 3y^2 dy - x^2 dz)$, the curve begins at $(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0)$ and ends at $(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$.
   b. Given a smooth curve $C \subset \mathbb{R}^n$, take the projections onto the coordinate axes $\{C \overset{\pi_i}{\to} Span(\hat{x}_i)\}$. Suppose these are diffeomorphisms onto their images. Prove: $\int_C \vec{F} \cdot \vec{C} = \sum \int_{\pi_i(C)} F_i(\pi_i^{-1}(x_i)) dx_i$. What is the orientation of $\pi_i(C)$ here?
   c. How to adjust the formula in b. when some projections are not bijective? (e.g. how to convert $\int_{S^1} \vec{F} \cdot d\vec{C}$ into $\int (..)dx + \int (..)dy$?) Using this formula recompute the integrals in part a.
   d. Let M be a one-dimensional manifold and assume $\int_M \omega$ exists. For a finite subset $S \subset M$ prove: $\int_M \omega = \int_{M \setminus S} \omega$.
   e. Take a neighborhood $S^1 \subset \mathcal{U} \subset \mathbb{R}^2$ and a function $f \in C^1(\mathcal{U})$. Define $\omega := df|_{S^1} \in \Omega^1(S^1)$. Compute $\int_{S^1} \omega$.

4. a. Prove: $V_1 \otimes \cdots \otimes V_k = 0$ iff at least one of V_j is a zero vector space.
   b. Establish the natural (basis-free) isomorphisms: i. $V_1 \otimes (V_2 \otimes V_3) \cong (V_1 \otimes V_2) \otimes V_3$.
   ii. $V \otimes (U \oplus W) \cong (V \otimes U) \oplus (V \otimes W)$. iii. $V^* \otimes W^* \cong (V \otimes W)^* \cong Hom(V, W^*) \cong Hom(W, V^*)$.
   c. Fix some bases: $V = Span\{v_i\}$, $W = Span\{w_i\}$.
      i. Prove: $\{v_i \otimes w_j\}$ is a basis of $V \otimes W$. (The non-trivial part here is the linear independence. Hint: use the dual bases of V^*, W^*.)
      ii. Write explicit bases for $Sym^k V^*$, $\overset{k}{\Lambda} V^*$. (Verify that these are bases.)
   d. Prove: the (skew-)symmetrization operations $Sym : \otimes^k V \to \otimes^k V$, $Alt : \otimes^k V \to \otimes^k V$ are projectors. In particular they result in (skew-)symmetric forms.
   e. Prove: i. $Sym(f \otimes g) = Sym(f \otimes Sym(g)) = Sym(Sym(f) \otimes g)$.
      ii. $Alt(f \otimes g) = Alt(f \otimes Alt(g)) = Alt(Alt(f) \otimes g)$.
   f. i. For some vectors $\{v_i\}$ in V prove: $v_1 \wedge \cdots \wedge v_k = 0$ iff these vectors are linearly dependent.
      ii. Verify: $f \wedge f = 0$ for any $f \in \overset{k}{\Lambda} V^*$ with k-odd.
      iii. Given $f_i \in \overset{k_i}{\Lambda} V^*$, verify: $f_1 \wedge \cdots \wedge f_k = \frac{(k_1 + \cdots k_f)!}{k_1! \cdots k_f!} Alt(f_1 \otimes f_k)$.