Geometric Calculus 2, 201.1.1041

Homework 7

Spring 2022 (D.Kerner)

Questions to submit: 1.b.i. 1.c. 1.d. 2.b. 2.d. 3.b. 3.c. 3.f. 4.b.

Some of the questions below were done in the class.

1. a. To an element $l \in V^*$ associate the linear function $l(x) \in \mathbb{R}[x]_1$ (homogeneous polynomials of degree=1). Accordingly we define the map $Sym^k V^* \ni Sym(l_1 \otimes \cdots \otimes l_k) \to l_1(x) \cdots l_k(x) \in \mathbb{R}[\underline{x}]_k$.
   Prove: this extends to an isomorphism of commutative graded algebras, $\oplus_k Sym^k(V^*) \cong \mathbb{R}[\underline{x}]$.

   b. Let $f \in \overset{k}{\Lambda} V^*$ and $g \in \overset{l}{\Lambda} V^*$. Verify the basic properties of the exterior product:
   i. $(f \wedge g) \wedge h = f \wedge (g \wedge h)$. ii. $(f+g) \wedge h = f \wedge g + f \wedge h$. iii. $f \wedge g = (-1)^{k \cdot l} g \wedge f$.

   c. Take the exterior algebra $\Lambda V^* = \underset{k}{\oplus} \overset{k}{\Lambda} V^*$. Compute $dim_{\mathbb{R}}(\Lambda V^*)$.

   d. Given vectors $v_1, \dots, v_k \in V$ and 1-forms $l_1, \dots, l_k \in V^*$ prove: $(l_1 \wedge \cdots \wedge l_k)(v_1 \otimes \cdots \otimes v_k) = det\{l_i(v_j)\}_{i,j}$.

2. a. Prove: any endomorphism $\phi \circlearrowright V$ induces the natural endomorphisms $\overset{k}{\Lambda} \phi \circlearrowright \overset{k}{\Lambda} V$ for $k = 0, 1, \dots$.
   (The maps $\overset{k}{\Lambda} \phi$ are defined without fixing a basis.)

   b. In particular, the map $\overset{n}{\Lambda} \phi \circlearrowright \overset{n}{\Lambda} V$ is the multiplication by a scalar, denote it $f(\phi)$.
   Prove: $f(\phi) = det[\phi]$ where $[\phi]$ is the presentation matrix of ϕ in some basis of V.

   c. Deduce: $det[\phi]$ does not depend on the choice of basis.
   Deduce: $det[U \cdot A \cdot U^{-1}] = det[A]$ for any $A \in Mat_{n \times n}(\mathbb{R})$ and $U \in GL(n, \mathbb{R})$.

   d. Given two endomorphisms, $\phi_1, \phi_2 \circlearrowright V$, study the scalar $f(\phi_1 \circ \phi_2)$. Deduce: $det[A \cdot B] = det[A] \cdot det[B]$.

   e. Given a homomorphism $\phi : V \to W$ and the corresponding homomorphisms $\overset{k}{\Lambda} \phi : \overset{k}{\Lambda} V \to \overset{k}{\Lambda} W$, construct the dual maps $\overset{k}{\Lambda} \phi^* : \overset{k}{\Lambda} W^* \to \overset{k}{\Lambda} V^*$.

   f. Verify: $(l_1 \wedge \cdots \wedge l_k)\big((\overset{k}{\Lambda} \phi)(v_1 \wedge \cdots \wedge v_k)\big) = \Big((\overset{k}{\Lambda} \phi^*)(l_1 \wedge \cdots \wedge l_k)\Big)(v_1 \wedge \cdots \wedge v_k)$. Deduce: $det[A] = det[A^t]$.

3. a. For $\omega_k \in \Omega^k(M)$ and $\omega_l \in \Omega^l(M)$ and a morphism $M \overset{\psi}{\to} \tilde{M}$ prove: $\psi^*(\omega_k \wedge \omega_l) = \psi^*(\omega_k) \wedge \psi^*(\omega_l)$.
   (We did the case $k = 1 = l$ in the class.)

   b. Compute $\phi^* \omega$ for $\omega = z dx \wedge dy + y dz \wedge dx \in \Omega^2(\mathbb{R}^3)$ and $\mathbb{R}^2 \ni (u, v) \overset{\phi}{\to} (cos(u), sin(u), v) \in \mathbb{R}^3$.

   c. Express the restriction of $dy \wedge dz|_{S^2}$ in polar coordinates on S^2.

   d. For $\Omega^k(\mathcal{U}) \overset{d}{\to} \Omega^{k+1}(\mathcal{U})$ prove: $d \circ d = 0$, i.e. $d(d\omega) = 0$ for every $\omega \in \Omega^k(\mathcal{U})$.

   e. We have defined the map $\Omega^k(M) \overset{d}{\to} \Omega^{k+1}(M)$ by a system of axioms. Prove: these define d uniquely.

   f. Given a map $\phi : M \to \tilde{M}$ and a form $\omega \in \Omega^k(\tilde{M})$ prove: $\phi^*(d\omega) = d(\phi^* \omega)$.

4. Suppose a submanifold $M \subset \mathbb{R}^{n+r}$ is defined by the equations $g_1(\underline{x}) = 0 = \cdots = g_r(\underline{x})$.
   a. (Dis)Prove: $dim(M) \geq n$.
   b. Suppose $dg_1, \dots, dg_r$ are linearly independent at each point of M. Take a function $f \in C^r(M)$. Prove: if $f|_M$ has a local extremum at $p \in M$ then $(df \wedge dg_1 \wedge \cdots \wedge dg_r)|_p = 0$. (This is Lagrange's theorem on conditional extremum)