Geometric Calculus 2, 201.1.1041

Homework 8

Spring 2022 (D.Kerner)

Questions to submit: 1.b. 1.e. 2.a. 2.d. 2.f. 2.g. 3.a. 3.b. 4.a.

The notion "the set X of dimension$\leq (n-1)$" was defined in the class.

1. a. Given a parameterized submanifold $\mathbb{R}^n_u \supseteq \mathcal{U} \stackrel{\phi}{\to} M \subset \mathbb{R}^N_x$ we have the tangent subspaces $T_pM \subset \mathbb{R}^N$. Take the standard inner product on $\mathbb{R}^N$ and restrict it to T_pM. Compute the angle between the vectors $\phi_*\hat{e}_i$, $\phi_*\hat{e}_j$ (for the standard basis of $\mathbb{R}^n$ at $\phi^{-1}(p)$).
   b. Vector fields ξ, η on $S^2 \setminus \{\pm\hat{z}\} \subset \mathbb{R}^3$ are of unit length, go along the parallels/meridians, and ξ is north-oriented, while η is west-oriented. Write the formulas for ξ, η in polar coordinates. Compute $[\xi,\eta]$.
   c. For $M_1 \stackrel{\psi_1}{\to} M_2 \stackrel{\psi_2}{\to} M_3$ and the corresponding pullbacks of Ω^k prove: $\psi_1^* \circ \psi_2^* = (\psi_2 \circ \psi_1)^*$.
   d. Deduce: if $\psi \circlearrowright M$ is smoothly invertible then all the pullbacks $\psi^* \circlearrowright \Omega^k(M)$ are (linear) isomorphisms.
   e. Take an embedding $\mathbb{R}^2_{xy} \supseteq \mathcal{U} \stackrel{\phi}{\hookrightarrow} \mathbb{R}^3_{xyz}$. Verify: $\phi^*(dy \wedge dz) = 0$, $\phi^*(dx \wedge dz) = 0$. Do this in as many ways as possible, e.g. by checking the action on $T_p\mathcal{U}$, by taking the pullback, by using $\phi^* d(\dots) = d\phi^*(\dots)$.

2. For a parameterized manifold we have defined $\int_M f dM = \int \phi^*(f) \cdot \sqrt{det[(\frac{\partial x}{\partial u})^t \cdot \frac{\partial x}{\partial u}]} \cdot du_1 \dots du_n$.
   a. Prove the linear algebra identity: $det[\mathbb{I} + \{v_i \cdot v_j\}_{ij}] = 1 + \sum v_i^2$.
   b. For $M \subset \mathbb{R}^3$ with a parametrization $\underline{x}(s,t)$ verify: $\int_M f dM = \int f(\underline{x}(s,t)) \cdot \|\partial_s \underline{x} \times \partial_t \underline{x}\| \cdot dsdt$. (Here $\partial_s \underline{x} \times \partial_t \underline{x}$ is the vector product.)
   c. Suppose $dim(M) = n$ and the subset $X \subset M$ is of dimension$\leq (n-1)$. Prove: $\int_M f dM = \int_{M \setminus X} f dM$. (Compare to the dim=1 case.)
   d. Suppose $M \subset \mathbb{R}^{n+1}$ is the graph of a function, i.e. $x_{n+1} = g(x_1, \dots, x_n)$. Prove: $\int_M f dM = \int_{\mathcal{U}} f \cdot \sqrt{1 + \|\nabla g\|^2} \cdot dx_1 \dots dx_n$.
   e. The hypersurface $M \subset \mathbb{R}^{n+1}$ is defined by the equation $h(x) = 0$. Suppose the projection $M \to \mathbb{R}^n_{x_1 \dots x_n}$ is injective and $\partial_{x_{n+1}} h$ does not vanish on $M \setminus X$, where $dim(X) \leq n-1$. Prove: $\int_M f dM = \int f \cdot \frac{\|grad(h)\|}{|\partial_{x_{n+1}} h|} dx_1 \dots dx_n$.
   f. Suppose M lies in a hyperplane $L \subset \mathbb{R}^{n+1}$, with $dim(M) = dim(L) = n$. Denote the angle between L and $\mathbb{R}^n_{x_1 \dots x_n}$ by α. Take the orthogonal projection $\mathbb{R}^{n+1} \to \mathbb{R}^n_{x_1 \dots x_n}$. Prove: $vol_n(M) \cdot cos(\alpha) = vol_n \pi(M)$.
   g. Take the torus of question 3.a from homework 2. For the parametrization of $\mathcal{U} \to M \subseteq S^1 \times S^1$ by two angles prove: $\int_M f dM = \iint_{(\phi_1,\phi_2) \in \mathcal{U}} f(\underline{x}(\phi_1, \phi_2)) \cdot r \cdot (R + r\sin(\phi_1)) d\phi_1 d\phi_2$. In particular, compute the surface area of the torus.

3. a. Take a vector subspace $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ and a vector $\xi \in \mathbb{R}^{n+1} \setminus \mathbb{R}^n$. Take two bases, $\{v_i\}$ and $\{w_j\}$ on $\mathbb{R}^n$. Prove: the bases $\{v_i\}, \{w_j\}$ have compatible orientations iff the bases $\{v_i\}, \xi$ and $\{w_j\}, \xi$ have compatible orientations.
   b. Prove: a smooth hypersurface $M \subset \mathbb{R}^{n+1}$ is orientable iff it is possible to chose unit normals $\{\hat{n}_p\}_{p \in M}$ that vary continuously with p.
   c. A subset $M \subset \mathbb{R}^{n+1}$ is defined by the equation $f(\underline{x}) = 0$. Suppose $grad(f)$ does not vanish at any point of M. Prove: M is an orientable manifold.

4. a. Take the standard charts $\mathbb{R}^n_+$, $\mathbb{R}^n_-$ on $S^n \subset \mathbb{R}^{n+1}$, with coordinate transformation $\underline{x} \to \underline{y} = \frac{x}{\|\underline{x}\|^2}$.
   Is this an oriented atlas? (See question 2.a.)
   b. Suppose $M \subset \tilde{M}$, with $dim(M) = dim(\tilde{M})$, and M is non-orientable. Prove: $\tilde{M}$ is non-orientable.
   Deduce: the Klein bottle and the real projective plane ($\mathbb{RP}^2$) are non-orientable.
   c. Take a manifold with two charts, $M = \mathcal{U}_1 \cup \mathcal{U}_2$, and assume $\mathcal{U}_1 \cap \mathcal{U}_2$ is path-connected.
   Prove: M is orientable. Deduce: S^n is orientable.
   d. Prove: if M is orientable and path connected, then there exist exactly two orientations.
   e. If M is orientable and has d connected components then the total number of possible orientations is 2^d.