Geometric Calculus 2, 201.1.1041

Homework 9

Spring 2022 (D.Kerner)

Questions to submit: 1. 2.e. 3.b. 4. 5.b. 5.c. 5.d. 5.e. 5.g.

The notion "the set X of dimension$\leq (n-1)$" was defined in the class.

1. For an embedded manifold $M \subset \mathbb{R}^N$ and $0 \leq k < \infty$ prove: $\Omega^k(\mathbb{R}^N) \twoheadrightarrow \Omega^k(M)$. (The case $k=1$ was in the class.)

2. Given two C^r-manifolds with their atlases, $(M_i, \mathcal{A}_i)$, consider the manifold $(M_1 \times M_2, \mathcal{A}_1 \times \mathcal{A}_2)$, whose charts are $\{\mathcal{U}^{(1)}_\alpha \times \mathcal{U}^{(2)}_\beta\}_{\alpha,\beta}$ and the coordinate maps are $\{\phi^{(1)}_\alpha \times \phi^{(2)}_\beta\}_{\alpha,\beta}$.
   a. Verify: $M_1 \times M_2$ is a C^r-manifold, and $dim(M_1 \times M_2) = dim(M_1) + dim(M_2)$.
   b. In the embedded case, $\psi_i : M_i \hookrightarrow \mathbb{R}^{N_i}$, verify: $\psi_1 \times \psi_2 : M_1 \times M_2 \hookrightarrow \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ is a manifold embedding.
   c. Give an example where N_1, N_2 are the minimal embedding dimensions for M_1, M_2, but $M_1 \times M_2$ can be embedded into $\mathbb{R}^{N_1+N_2-1}$.
   d. Prove: if M_1, M_2 are compact/path-connected/orientable, then so is $M_1 \times M_2$. (Deduce: $S^{n_1} \times \cdots S^{n_k}$ is orientable.)
   e. In the embedded oriented case take the volume forms, $\Omega_i \in \Omega^{n_i}(M_i)$. The candidate for the volume form on $M_1 \times M_2 \subset \mathbb{R}^{N_1+N_2}$ is "$\Omega_1 \times \Omega_2$". Formulate and prove the precise statement. In particular verify: $vol_{n_1+n_2}(M_1 \times M_2) = vol_{n_1}(M_1) \cdot vol_{n_2}(M_2)$.

3. a. Prove: the (non-)orientability of M depends only on the C^1-diffeomorphism type of M (and not on an embedding $M \hookrightarrow \mathbb{R}^N$).
   b. Prove: the real projective space $\mathbb{RP}^n$ (see hwk.4. q.3) is orientable iff n is odd. (Hint: use the covering $S^n \to \mathbb{RP}^n$.)
   c. Suppose $dim(M) = n$ and there exists a form $\omega \in \Omega^n(M)$ without zeros, i.e. $\omega|_p \neq 0 \in \overset{n}{\Lambda} T^*_p M$ for each $p \in M$. Prove: M is orientable.

4. Compute the area of the surface $S = \{(x,y,z) | x^2+y^2+z^2 = a^2,\ \frac{x^2}{a^2} + \frac{z^2}{b^2} \leq 1\}$, here $0 < b \leq a$.

5. a. Take a parameterized hypersurface, $\mathbb{R}^n \supset \mathcal{U} \to M \subset \mathbb{R}^{n+1}$, $\underline{u} \to \underline{x}(\underline{u})$. (The orientation of M is induced from that of $\mathcal{U}$.) Prove: the flux of a vector field $\vec{F}$ through M equals $\int_\mathcal{U} det\left[\vec{F}, \partial_{u_1}\underline{x}, \ldots, \partial_{u_n}\underline{x}\right] du_1..du_n$. (We have proved this in the class.)
   b. Convert this into an explicit formula in the particular case of the graph of a function, i.e. $M = \{(x,y,z) | z = z(x,y)\} \subset \mathcal{U}_{xy} \times \mathbb{R}^1_z$. Verify that the orientation of $\mathcal{U}$ is compatible with the upper normal to M.
   c. Suppose $M_{dim=2} \subseteq S^2 \subset \mathbb{R}^3$, with the outer normal. Given a field $\vec{F} = f \cdot \vec{r}$ prove: its flux is $\iint_S \vec{F} d\vec{S} = \iint_\mathcal{U} f \cdot r^3 sin(\theta) d\theta d\phi$. (Which order of θ, ϕ corresponds to the outer normal of S^2?) In particular, compute the flux of $\vec{F} = \frac{\vec{r}}{r^d}$ through $S^2 \subset \mathbb{R}^3$.
   d. Take a smooth surface $S \subset \mathbb{R}^3$. Suppose the projections π_x, π_y, π_z of S onto all the coordinate planes are C^1-diffeomorphisms onto their images. (Thus S is the graph of functions, $z = z(x,y)$, $y = y(x,z)$, $x = x(y,z)$.) Fix an orientation on S, i.e. choose the normal $\hat{n} = (n_x, n_y, n_z)$.
      - Verify: each of the functions n_x, n_y, n_z has a (locally) constant sign on S.
      - Let $\omega = f_x dy \wedge dz + f_y dz \wedge dx + f_z dx \wedge dy|_S \in \Omega^2(S)$. Prove: $\int_S \omega = \int_{\pi_z(S)} f(x,y,z(x,y)) \cdot sign(n_z) \cdot dxdy + \int_{\pi_x(S)} f(x(y,z),y,z) \cdot sign(n_x) \cdot dydz + \cdots$.
   e. Compute $\int_S \omega$ where $\omega = (x+z)x \wedge dy + (z+y+cos(x))dy \wedge dz + (x - sin(y))dz \wedge dx|_S$. Here $S = (\partial Pyramid) \setminus \mathcal{U}$, with $Pyramid \subset \mathbb{R}^3$ defined by $x,y,z \geq 0$, $x+y+z \leq 1$, and $\mathcal{U} \subset \mathbb{R}^2_{xy}$ is defined by $x,y \geq 0$, $x^2+y^2 \leq \frac{1}{\sqrt{2}}$. (The orientation corresponds to the outer normal.)
   f. Compute $\int_C \vec{F} \cdot d\vec{C}$ in the following cases: i. $\vec{F} = \frac{(-y,x)}{x^2+y^2}$, $C = \{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\} \subset \mathbb{R}^2$ (counterclockwise). ii. $\vec{F} = \frac{(-y,x)}{x^2+y^2}$, and the curve $(\sqrt{3},1) \rightsquigarrow (-\sqrt{3},1)$ is given in polar coordinates by $r(\theta) = \frac{1}{1-sin(\theta)}$.
   g. For each $n \in \mathbb{N}$ give a closed oriented curve that does not pass through $(0,0)$ and satisfies: $\oint_C \frac{(-y,x)}{x^2+y^2} d\vec{C} = 2\pi n$.