Introduction to Commutative Algebra

201.1.7071, Fall 2022, (D.Kerner)

Homework 0

(to be fully solved before the first lecture)

Commutative algebra is "even more abstract" than linear algebra. It is hard to work with rings/ideals/modules without a pre-acquired minimal package of examples (and the corresponding intuition). This is the current goal.

All our rings will be commutative, unital (with $0 \neq 1$). $\Bbbk$ denotes a(ny) field. R denotes a(ny) ring. $\Bbbk[[\underline{x}]] := \Bbbk[[x_1, \ldots, x_n]]$ is the ring of formal power series. Given an open subset $\mathcal{U} \subseteq \mathbb{R}^n$ we denote by $C^r(\mathcal{U})$ the ring of functions on $\mathcal{U}$ that are r-times continuously differentiable. (Here $0 \leq r \leq \infty$.) Denote by $C^\omega(\mathcal{U})$ the ring of functions real-analytic on $\mathcal{U}$. For an open subset $\mathcal{U} \subseteq \mathbb{C}$ denote by $\mathcal{O}(\mathcal{U})$ the ring of function complex-analytic on $\mathcal{U}$. Given an ideal $I \subsetneq R$ we take the quotient ring ${}^{R}/_{I}$.

Questions in *cursive* are intentionally imprecise.

1. a. Write down the definition of a ring, of an ideal, of a homomorphism of rings.
   b. Given a ring R and indeterminates $\underline{x} = \{x_1 \ldots x_n\}$ write the definition of the rings $R[\underline{x}]$, $R[[\underline{x}]]$.
   c. (Dis)Prove: i. $R[\underline{x}] = R[x_1][x_2] \ldots [x_n]$. ii. $R[[\underline{x}]] = R[[x_1]][[x_2]] \ldots [[x_n]]$. iii. $\Bbbk[x_1][[x_2]] = \Bbbk[[x_2]][x_1]$.
   d. Denote by $R^\times \subset R$ the subset of all the invertible elements. Prove: $R^\times$ is an abelian group. (What is the group operation?)
   e. Describe (explicitly) $R^\times$ for the following rings: i. $\Bbbk[\underline{x}]$ ii. $\Bbbk[[\underline{x}]]$ iii. $C^r(\mathcal{U})$, $0 \leq r \leq \infty, \omega$.
   f. Prove: any element of $\Bbbk[[x]]$ is presetable as $u \cdot x^d$. Here $u \in R^\times$ and $d \in \mathbb{N}$ are uniquely defined.
   g. Let $A \in Mat_{n \times n}(R)$. Prove: A is invertible iff $det(A) \in R^\times$.
   Give examples with $R = \mathbb{C}[x]$ where $det(A)$ has no zeros in $\mathbb{C}$ but A is non-invertible.

2. a. Write down the definition of an integral domain. (We will call this just "a domain".)
   b. When are the rings $R[\underline{x}]$, $R[[\underline{x}]]$ domains? (Give a simple necessary and sufficient condition.)
   c. Suppose $\mathcal{U} \subseteq \mathbb{C}$ is connected. Is $\mathcal{O}(\mathcal{U})$ a domain?
   d. ($0 \leq r \leq \infty$) Prove: $f \in C^r(\mathcal{U})$ is a zero divisor iff the zero locus $f^{-1}(0) \subseteq \mathcal{U}$ has a non-empty interior.
   e. Describe (explicitly) the invertible elements of $\Bbbk[x_1 \ldots, x_n]$, $\Bbbk[[x_1 \ldots, x_n]]$, $C^r(\mathcal{U})$, $0 \leq r \leq \infty$.

3. a. Given an ideal $I \subset R$ write down the definition of the quotient ring ${}^{R}/_{I}$.
   b. Let $1 \leq d < \infty$. Define the map $Taylor_d : C^\infty(-1, 1) \to {}^{\mathbb{R}[x]}/_{(x)^{d+1}}$ by Taylor-expanding (at 0) up to order d. Prove: $Taylor_d$ is a surjective homomorphism of rings. What is its kernel?
   c. Define the map $Taylor : C^\infty(-1, 1) \to \mathbb{R}[[x]]$ by taking the full Taylor expansion at 0. What is the kernel of this homomorphism?
   Borel's lemma: this homomorphism is surjective.
   d. Given a homomorphism of rings, $\phi : S \to R$, prove: $ker(\phi) \subset S$ is an ideal. Prove: ϕ factorizes into $S \to {}^{S}/_{ker(\phi)} \to R$.

4. a. Prove: any homomorphism $\Bbbk \to R$ is injective.
   b. Prove: there exists (and unique) homomorphism of rings $\mathbb{Z} \to R$. Prove: its kernel is the ideal of the form $(n) \subset \mathbb{Z}$, where $n =: char(R)$ is the *characteristic of* R.
   c. Compute $char(R)$ for the following rings.
   i. $char\, {}^{\mathbb{Z}[\underline{x}]}/_{(n)}$ for a number $n \in \mathbb{Z}$ ii. $\Bbbk[x]$ iii. $\Bbbk[[x]]$ iv. $C^r(\mathcal{U})$ v. $\mathcal{O}(\mathcal{U})$.
   d. For any ideal I prove: $char\, {}^{\Bbbk[\underline{x}]}/_{I} = char(\Bbbk)$ and $char\, {}^{\Bbbk[[\underline{x}]]}/_{I} = char(\Bbbk)$.

5. a. Let $\mathfrak{m}_p \subset \Bbbk[x,y]$ be the set of all polynomials vanishing at the point $p \in \Bbbk^2$. Prove: $\mathfrak{m}_p$ is an ideal with two generators. Write a couple of generators.
   b. Fix three (distinct) points lying on a line in $\Bbbk^2$. Let $I \subset \Bbbk[x,y]$ be the ideal of all the polynomials vanishing at these points. Prove: $I = \langle l, q_3 \rangle$, where l is a linear form, while $(l) \not\ni q_3$ is a cubic polynomial. (Hint: apply the action $GL(2,\Bbbk) \circlearrowright \Bbbk^2$ to bring these points to a nice position.)
   What happens for three points not on one line?
   c. Let $\Bbbk = \bar{\Bbbk}$. List (explicitly) all the maximal ideals of $\Bbbk[x]$. (Deduce: these maximal ideals correspond to the points of the line $\Bbbk^1$.) Prove: any non-zero prime ideal in $\Bbbk[x]$ is maximal.
   d. Give examples of maximal ideals in $\mathbb{R}[x]$ that are not of the type you have seen in 4.c. (Can you explain this geometrically?)
   e. Take two ideals, $I = \langle x, y \rangle \subset \Bbbk[x,y]$ and $J = \langle x-1, y-1 \rangle \subset \Bbbk[x,y]$. Prove: $I \cap J \supsetneq I \cdot J$. (What is the geometric meaning of $I \cap J$ and $I \cdot J$?)
   f. Give (many) examples of prime ideals in $\Bbbk[x,y]$ that are not maximal. *What is the geometry?*
   g. List all the maximal ideals of $\mathbb{Z}$. *What is the geometry?*
   h. Prove: the rings $\Bbbk[[\underline{x}]]$, $\Bbbk[[\underline{x}]]/J$ have exactly one maximal ideal. *What is the geometry?*
   i. Prove: i. $\Bbbk[x]/(x^2-1) \cong \Bbbk[x]/(x-1) \times \Bbbk[x]/(x+1) \cong \Bbbk \times \Bbbk$. ii. $\Bbbk[x]/(x-1)^2 \not\cong \Bbbk \times \Bbbk$.
   iii. (for $\Bbbk = \bar{\Bbbk}$) $\Bbbk[x,y]/(x^n-1, y^m-1) \cong \underbrace{\Bbbk \times \cdots \times \Bbbk}_{mn}$. *What is the geometry?*

6. a. The *radical* of an ideal $I \subset R$ is the subset $\sqrt{I} := \{r \in R | \ r^d \in I \text{ for } d \gg 1\}$. Prove: $\sqrt{I} \subset R$ is an ideal.
   b. Take $I = \langle x^2+y^2-1, x^2-1 \rangle \subset \Bbbk[x,y]$. Compute $\sqrt{I}$. (What is the geometry?)
   c. Suppose $I \supseteq J \supseteq I^d$ for some $d \in \mathbb{N}$. Prove: $\sqrt{I} = \sqrt{J}$.
   d. Let $\mathfrak{m} \subset C^\infty(\mathbb{R}^n)$ be the ideal of functions vanishing at the origin. What are its generators?
   e. Denote by $\mathfrak{m}^\infty \subset C^\infty(\mathbb{R}^n)$ the subset of functions "flat" at $o \in \mathbb{R}^n$. (i.e. their derivatives of all orders vanish at o. e.g. $e^{-\frac{1}{\|x\|^2}}$) Prove: $\mathfrak{m}^\infty = \cap_{d \in \mathbb{N}} \mathfrak{m}^d$. Compute $\sqrt{\mathfrak{m}^\infty}$.
   f. An ideal $I \subset R$ is called radical if $\sqrt{I} = I$. (Dis)prove: i. Any radical ideal of $\mathbb{Z}$ is generated by a prime number. ii. Any radical ideal of $\Bbbk[x,y]$ has at most two generators. (Hint: 5.b.)
   g. Give (many) examples of rings with nilpotents. Prove: the ring $C^r(\mathcal{U})$ has no nilpotents.
   h. Prove: if $x \in R$ is nilpotent then $1-x$ is invertible in R.
   i. Define the *nilradical* of R as $nil(R) := \sqrt{(0)}$. Prove: $\sqrt{(0)}$ is the union of all the nilpotent elements.
   j. Prove: $nil(R/I) = \sqrt{I}/I$. In particular, the ring $R/nil(R)$ has no nilpotents. Is the ring $R/nil(R)$ necessarily an integral domain?

7. (Division with remainder)
   a. Prove: for any $a, b \in \mathbb{N}$ there exists the unique presentation $a = bq + r$, where $q \in \mathbb{N}$ and $0 \le r \le b-1$.
   b. Prove: for any $a, b \in \Bbbk[x]$ there exists the unique presentation $a = bq + r$, where $q \in \Bbbk[x]$ and $deg_x(r) \le deg_x(b) - 1$.
   c. Conclude: every ideal in $\Bbbk[x]$ is a principal ideal (i.e. can be generated by one element.)

8. Please read Chapter.0 of M.Reid's "Undergraduate Commutative Algebra".