Introduction to Commutative Algebra

201.1.7071, Fall 2022, (D.Kerner)

Homework 1

Submission date: 17.11.2022 (preferably by e-mail)
Questions to submit: 1.d. 1.f. 2.a.ii. 2.a.iv. 3.c. 3.e. 4.c. 5.c.

Below I, J are ideals, $\mathfrak{p}$ is a prime ideal, $\mathfrak{m}$ is a maximal ideal. $R[[\underline{x}]] := R[[x_1, \dots, x_n]]$.

1. a. Prove: $(0) \subset R$ is a maximal ideal iff R is a field.
   b. Describe all the nilpotents in the ring $\Bbbk[x]/_{(f)}$, where $f(x) = \prod(x - x_i)^{d_i}$, $x_i \neq x_j$ and $d_i \geq 1$.
   c. Prove: the ring $R[[\underline{x}]]$ is a domain iff R is a domain.
   d. Describe all the quotient rings of $\Bbbk[[x]]$, $n = 1$ (up to isomorphism).
   If the quotient is a finite dimensional vector space, give a basis.
   e. We have introduced the induced topology on $V(I) \subset \mathfrak{m}Spec(R)$. Verify: this is a topology.
   f. We have partially established the homeomorphism $\mathfrak{m}Spec(R/_I) \cong V(I) \subset \mathfrak{m}Spec(R)$.
   Write the proof in details.

2. a. Given a morphism of rings $\phi : R \to S$. (Dis)Prove:
   i. $\phi(0) = 0$.
   ii. $\phi(I) \subset S$ is an ideal for any ideal $I \subset R$.
   iii. $\phi^{-1}(I) \subset R$ is an ideal for any ideal $I \subset S$.
   iv. If $I \subset S$ is a prime/maximal ideal then so is $\phi^{-1}(I) \subset R$.
   b. Suppose a statement in 2.a. is false, does it become true if ϕ is injective/surjective?

3. a. For a set of ideals $\{I_\lambda\}_{\lambda \in \Lambda}$ write down the definitions of $\cap_{\lambda \in \Lambda} I_\lambda$ and $\sum_{\lambda \in \Lambda} I_\lambda$.
   Assuming Λ is finite, write down the definition of $\prod_{\lambda \in \Lambda} I_\lambda \subset R$. Verify that all these are ideals.
   b. Prove: $\cap_{\lambda \in \Lambda} V(I_\lambda) = V(\sum_{\lambda \in \Lambda} I_\lambda) \subset \mathfrak{m}Spec(R)$.
   c. For a finite Λ prove: $\cup_{\lambda \in \Lambda} V(I_\lambda) = V(\prod_{\lambda \in \Lambda} I_\lambda) \subset \mathfrak{m}Spec(R)$.
   d. (Dis)Prove: i. $I \cdot J = I \cap J$. ii. If I, J are primes then so is $I + J$.
   iii. $I \cup J \subset R$ is an ideal iff $I \subseteq J$ or $J \subseteq I$. (It is worth to use the geometry, 3.b.)
   e. Given two primes $\mathfrak{p}_1, \mathfrak{p}_2 \subset R$ is $\mathfrak{p}_1 \cap \mathfrak{p}_2$ a prime ideal?
   f. Let $S \subset R$ a multiplicative set. Is $R \setminus S$ an ideal?
   g. For two ideals $I, J \subset R$ and a prime $\mathfrak{p} \subset R$ prove: $I \cdot J \subseteq \mathfrak{p}$ iff $I \cap J \subseteq \mathfrak{p}$ iff ($I \subseteq \mathfrak{p}$ or $J \subseteq \mathfrak{p}$).
   What is the geometric interpretation?

4. a. Prove: the ring $R/_{nil(R)}$ is reduced.
   b. Prove: $\mathfrak{m}Spec(R) \cong \mathfrak{m}Spec(R/_{nil(R)})$.
   c. Establish the universal property: any homomorphism $R \to S$, with S-reduced, factorizes uniquely into $R \to R/_{nil(R)} \to S$.
   d. Prove: $R^\times = R^\times + nil(R)$. (Namely, if u is a unit and x is nilpotent then $u + x$ is a unit.)

5. a. Let $R \cong R_1 \times R_2$ (the direct product of rings). Prove: R contains non-trivial idempotents.
   b. Prove: the (natural) projections $R_1 \overset{\pi_1}{\leftarrow} R_1 \times R_2 \overset{\pi_1}{\to} R_2$ are homomorphisms of rings.
   Does π_i admit a right inverse? (i.e. a homomorphism $R_i \overset{s_i}{\to} R_1 \times R_2$ satisfying: $\pi_i \circ s_i = Id_{R_i}$.)
   c. Establish the embedding $\phi : \Bbbk[x, y]/_{(xy)} \hookrightarrow \Bbbk[x] \oplus \Bbbk[y]$.
   d. (More generally) A prime $\mathfrak{p} \subset R$ is called minimal if there is no other prime $\mathfrak{p}' \subsetneq \mathfrak{p}$. Prove: if R is reduced and has only finitely many minimal primes, $\{\mathfrak{p}_i\}$, then $R \hookrightarrow \prod R/_{\mathfrak{p}_i}$.
   e. Use Zorn's lemma to prove: any prime contains a minimal prime.