Introduction to Commutative Algebra

201.1.7071, Fall 2022, (D.Kerner)

Homework 10

Not for submission.

By $A \sim B$ we denote the left-right equivalence of matrices, see q.3 of hwk.4. Recall: every PID is a UFD.

1. a. Let $A \in Mat_{n\times n}(R)$ and suppose $det(A) \in R$ is non-nilpotent. Denote by S the multiplicative group, generated by $det(A)$. The morphism $A : R^n \to R^n$ induces the morphism $S^{-1}A : S^{-1}R^{\oplus n} \to S^{-1}R^{\oplus n}$. Prove: $S^{-1}A$ is an injective morphism.
   b. Establish an isomorphism of $R_\mathfrak{p}$-modules: $({}^{R}\!/_{I})_\mathfrak{p} \cong {}^{R_\mathfrak{p}}\!/_{I_\mathfrak{p}}$, for any prime $\mathfrak{p} \subset R$.

2. Let M be a f.g. R-module.
   a. Prove: if R is Noetherian then M is finitely-presented.
   b. Prove: $ann(S^{-1}R \cdot M) = S^{-1}R \cdot ann(M) \subset S^{-1}R$ for any multiplicative set $0 \notin S \subset R$.
   c. Prove: $M = 0$ iff $M_\mathfrak{m} = 0$ for any maximal ideal $\mathfrak{m} \subset R$.
   (Hint: for the direction $\Leftarrow$ it is enough to prove: $ann(M) \not\subseteq \mathfrak{m}$ for any $\mathfrak{m} \subset R$.)
   d. Prove: $I = J \subset R$ (for two ideals) iff $I_\mathfrak{m} = J_\mathfrak{m} \subset R_\mathfrak{m}$ for every $\mathfrak{m} \subset R$.
   e. Prove: ${}^{R}\!/_{I} \cong {}^{R}\!/_{J}$ (isomorphism of R-modules) iff $I = J \subset R$.
   Show that this does not hold if ${}^{R}\!/_{I} \cong {}^{R}\!/_{J}$ is only an isomorphism of rings.
   f. Suppose R is Noetherian and $\mathfrak{m} \subset R$ a maximal ideal. Prove: $({}^{R}\!/_{I})_\mathfrak{m} \cong ({}^{R}\!/_{J})_\mathfrak{m}$ iff $S^{-1} \cdot I = S^{-1} \cdot J$, where the multiplicative set S is generated by a non-nilpotent element $g \in R$.

3. Let $(R, \mathfrak{m})$ be a local ring, let $M \in mod\text{-}R$, with a presentation $M = coker(A)$, for some $A \in Mat_{m\times n}(R)$.
   a. Suppose $\oplus {}^{R}\!/_{I_i} \cong \oplus {}^{R}\!/_{J_j}$ (as R-modules). Prove (after a permutation of indices): $I_1 = J_1 \subset R$, $I_2 = J_2 \subset R$, ... Show that this is not true over a non-local PID.
   b. Define the "elementary column-operations" on a matrix: $C_i \rightsquigarrow u \cdot C_i$ (for $u \in R^\times$); $C_i \leftrightarrow C_j$; $C_i \rightsquigarrow C_i + x \cdot C_j$ (for any $x \in R$). Prove: any $A \in GL_n(R)$ is a product of elementary matrices.
   c. i. Prove: A is a minimal presentation of M iff $Im(A) \subseteq \mathfrak{m} \cdot R^{\oplus m}$ and $ker(A) \subseteq \mathfrak{m} \cdot R^{\oplus n}$.
      ii. Prove: every finitely-presented module admits a minimal presentation. More precisely: $A \sim \mathbb{1} \oplus [\tilde{A}|\mathbb{O}]$, where $\tilde{A} \in Mat_{\tilde{m}\times\tilde{n}}(\mathfrak{m})$, and $ker(\tilde{A}) \subseteq \mathfrak{m} \cdot R^{\oplus \tilde{n}}$.
      iii. Prove: if A is a minimal presentation then $m = dim_{R/\mathfrak{m}} {}^{M}\!/_{\mathfrak{m}\cdot M}$. Moreover, m is the minimal number of generators of M (and is independent of choices).
      Prove: n is also well-defined.
      iv. We have proved in the class: the minimal presentation is unique (up to the left-right equivalence). Go over the details.

4. Let $A \in Mat_{m\times n}(R)$, with $m \le n$. The j'th determinantal ideal $I_j(A) \subseteq R$ is generated by the determinants of all $j \times j$-minors of A.
   a. Verify: $R =: I_0(A) \supseteq I_1(A) \supseteq \cdots \supseteq I_m(A) \supseteq I_{m+1}(A) := 0$.
   b. Compute $\{I_j(A)\}$ for the matrices: $\begin{bmatrix} x^3+6 & x^2-7 \\ x^2+7 & x^3-6 \end{bmatrix} \in Mat_{2\times 2}(\Bbbk[x])$, $\begin{bmatrix} \sum_{j=0}^{\infty} x^{j+2} & 1+x \\ \sum_{j=0}^{\infty}(-x^j) & 1-x \end{bmatrix} \in Mat_{2\times 2}(\Bbbk[[x]])$.
   c. Let $R = \Bbbk[\underline{x}]$, $\Bbbk = \bar{\Bbbk}$. Prove: $p \in V(I_j(A))$ iff $rank(A|_p) < j$. (Here $A|_p = A \otimes {}^{R}\!/_{\mathfrak{m}_p} \in Mat({}^{R}\!/_{\mathfrak{m}_p})$.)
   d. Prove: $I_j(UAV) = I_j(A)$ for any $U \in GL(m, R)$, $V \in GL(n, R)$.
   (Hint: it is enough to prove $I_j(UAV) \subseteq I_j(A)$. Reduce this to the local case, via q.2.d. Then use q.3.b.)

5. Let R-PID.

a. Given elements $\alpha_1, \alpha_2 \in R$, fix a generator $(\alpha) = (\alpha_1, \alpha_2) \subset R$. Prove: $\begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix} \sim \begin{bmatrix} \alpha & 0 \\ 0 & \frac{\alpha_1 \alpha_2}{\alpha} \end{bmatrix}$.
b. For any diagonal matrix prove: $diag[\alpha_1 \dots \alpha_m] \sim diag[\lambda_1 \dots \lambda_m]$, where $(\lambda_1) \supseteq (\lambda_2) \supseteq \cdots$.
c. Prove the corresponding statement for the module $\oplus R/_{(\alpha_i)}$.
d. Given coprime elements $\{b_i\}$ in R prove: $R/_{(\prod b_i^{p_i})} \cong \oplus R/_{(b_i^{p_i})}$ (isomorphism of R-algebras).
e. Express the invariants factors $\{(\lambda_i)\}$ of A via the ideals $\{I_j(A)\}$.
In particular, the ideals $\{(\lambda_i)\}$ are uniquely-defined.
f. Define a morphism $\phi : \mathbb{Z}^3 \to \mathbb{Z}^4$ by $(a, b, c) \to (4a - 3b, 6a + 2c, 10a + 6b + 8c, 3b + 2c)$. Write down the invariant and the primary decomposition of the module $coker[A] \in mod\text{-}\mathbb{Z}$.
g. Find the Smith normal form for the matrices of q.4.b.
h. Prove: $\mathbb{Q}$ is an indecomposable $\mathbb{Z}$-module.
Any contradiction to the structure theorem of modules over a PID?
i. Prove the uniqueness part of that structure theorem: $M_1 \cong M_2$ iff $rank(M_1) = rank(M_2)$ and their invariant factors coincide.