Introduction to Commutative Algebra

201.1.7071, Fall 2022, (D.Kerner)

Homework 2

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Questions to submit: 1.b. 1.d. 2.d. 2.f. 2.h.ii. 4.a. 4.b.ii. 4.b.iii. 5.a. (preferably by e-mail)

1. a. Take a (non-trivial) idempotent $e \in R$. Prove: the function corresponding to e is identically one on some connected components of $\mathfrak{m}Spec(R)$ and vanishes on the other components.
   b. Write down all the idempotents in the rings:
   i. $Z/_{(12)}$ ii. $\mathbb{k}[x]/_{((x-x_1)(x-x_2)(x-x_3))}$, here x_1, x_2, x_3 are all distinct.
   c. Compute the number of idempotents in the following rings: i. $\mathbb{Z}/_{(\prod p_i^{d_i})}$, here the primes $\{p_i\}$ are pairwise distinct. ii. $\mathbb{k}[x]/_{(\prod(x-x_i)^{d_i})}$, here the points $\{x_i\}$ are pairwise distinct.
   d. Suppose $e_1 \dots e_n \in R$ are idempotents. Prove: the ideal $(e_1, \dots, e_n) \subset R$ is principal and is generated by an idempotent. (Hint: how to orthogonalize the idempotents?)

2. a. (Dis)Prove: i. $R/_I[\underline{x}] \cong R[\underline{x}]/_{I \cdot R[\underline{x}]}$. ii. $R/_I[[\underline{x}]] \cong R[\underline{x}]/_{I \cdot R[[\underline{x}]]}$.
   b. Suppose $\{I \subseteq J_\alpha\}_{\alpha \in A}$ for some ideals $I, \{J_\alpha\}$. Prove: $\cap[J_\alpha] = [\cap J_\alpha] \subseteq R/_I$.
   Disprove "$\cap[J_\alpha] = [\cap J_\alpha]$" if one does not assume $\{I \subseteq J_\alpha\}_{\alpha \in A}$.
   c. We have proved in the class: $\sqrt{I} = \cap_{I \subseteq \mathfrak{p}} \mathfrak{p}$. Go over all the details. Disprove: $\sqrt{I} = \cap_{I \subseteq \mathfrak{m}} \mathfrak{m}$.
   d. Suppose $I, J \subset R$ are radicals. (Dis)Prove: i. $I \cap J$ is radical. ii. $I + J$ is radical.
   e. Prove: all the primes of the ring $R_1 \times R_2$ are of the form $\mathfrak{p}_1 \times R_2$ or $R_1 \times \mathfrak{p}_2$.
   f. For any ring prove: $R = R^\times \coprod(\cup_{\mathfrak{m} \subset R} \mathfrak{m})$ (the union over all the maximals). Geometry: given $f \in R$, if the associated "function" $\mathfrak{m}Spec(R) \to \dots$ has no zeros, then f is invertible.
   g. Define the "Jacobson radical" of a ring as $rad(R) := \cap \mathfrak{m}$ (the intersection of all the maximals).
   Compute $rad(R)$ for: i. $\mathbb{k}[\underline{x}]$ ii. $\mathbb{k}[[\underline{x}]]$ iii. $\mathbb{k}[[x]] \times \mathbb{k}[[y]]$ iv. $R_1 \times R_2$.
   h. Prove: i. $rad(R) \supseteq nil(R)$. ii. $R^\times + rad(R) = R^\times$. iii. If $R^\times + I = R^\times$ then $I \subseteq rad(R)$.

3. a. Prove: i. $(a) = (b)$ iff $a|b$ and $b|a$. ii. For R a domain: $(a) = (b)$ iff $a = ub$ for some $u \in R^\times$.
   iii. Disprove ii. over non-domains. iii. If $a|b$ and $(a) = (a')$, $(b) = (b')$ then $a'|b'$.
   b. Give an example of a prime but non-irreducible, and an example of an irreducible but non-prime.

4. a. Prove: the ring $C^\omega(-\epsilon, \epsilon)$ is non-local for any $\epsilon > 0$.
   b. Suppose R is local. (Dis)Prove: i. $R/_I$ is local. ii. $R[x]$ is local. iii. $R[[x]]$ is local.
   c. Describe explicitly the rings: i. $\mathbb{k}[x,y]_{(x,y)}$ ii. $\mathbb{k}[x,y]_{(x)}$. (What is the geometry?)
   d. (For those who learned Complex Functions) Prove: the ring $\mathbb{C}\{\underline{x}\}$ is local. (In fact $\mathbb{C}\{x\} = \mathbb{C}\{x\}^\times + (x)$.) Deduce: the ring $\mathbb{R}\{x\}$ is local.

5. a. Write the (most natural) basis of the $\mathbb{k}$-vector space $\mathbb{k}[x]$. Is this a basis of the space $\mathbb{k}[[x]]$? Prove: $\mathbb{k}[[x]]$ is not countably generated over $\mathbb{k}$.
   b. We have stated and partially proved several "naturality/universality" properties of the extension $R \hookrightarrow Frac(R)$. Write down all the proofs.
   c. For a domain R take the field of fraction $R \hookrightarrow Frac(R))$ and a polynomial $f \in R[x]$.
      i. Suppose $\frac{b}{c} \in Frac(R)$ is a root of $f(x) = a_0 + \dots + a_n x^n \in R[x]$, and moreover $(bc) = (b) \cdot (c)$. Prove: $a_0 \in (b)$ and $a_n \in (c)$.
      ii. For R-UFD prove: if $f(x) \in R[x]$ is monic then it has a root in R iff it has a root in $Frac(R)$.