Introduction to Commutative Algebra

201.1.7071, Fall 2022, (D.Kerner)

Homework 3

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Questions to submit: 1. 3.a. 3.c. 4.a. 4.b. 6.a. 6.c. 7.b. (preferably by e-mail)

1. Are the rings $\mathbb{k}[x,y]/(x^2+y^2-1)$ and $\mathbb{k}[z]$ isomorphic?

2. (Chinese remainder theorem) Ideals $I, J \subset R$ are called co-maximal if $I + J = R$. (Give examples.)
   a. Suppose the ideals $I_1, \dots, I_n$ are pairwise co-maximal. Prove: i. The ideals $I_1, (I_2 \cdots I_n)$ are co-maximal. ii. $\cap I_j = \prod I_j$. iii. $R/\prod I_j \cong \prod R/I_j$. (Hint: start from $n = 2$.)
   b. Let R be a PID. Prove: the elements $x, y \in R$ are coprime (i.e. share no common prime factors) iff the ideals $(x), (y) \subset R$ are co-maximal.

3. a. List the minimal prime ideals in the ring $\mathbb{k}[x]/(f)$, with $\mathbb{k} = \bar{\mathbb{k}}$.
   b. Prove: R is a domain iff (0) is the only minimal prime ideal.
   c. Suppose R has a finite number of minimal primes, $\mathfrak{p}_1, \dots, \mathfrak{p}_n$. Take the (natural) projection $\phi : R \to \prod R/\mathfrak{p}_i$ Prove: $ker(\phi) = nil(R)$.
   Prove: $\phi(R)$ intersects non-trivially all the ideals of the type $0 \times \cdots \times 0 \times R/\mathfrak{p}_i \times 0 \times \cdots \times 0$.

4. a. Prove: R is local iff all the non-units form an ideal iff exists $\mathfrak{m} \subset R$ s.t. $1 + \mathfrak{m} \subseteq R^\times$.
   b. Prove: $\mathbb{Z}_{(p)}/(p) \cdot \mathbb{Z}_{(p)} \xrightarrow{\sim} \mathbb{Z}/(p)$.

5. Let $\{M_i\}$ be submodules of M. Prove: $M = \oplus M_i$ iff one of the following holds:
   a. $M = \sum M_i$, and the presentation $m = \sum m_i$ is unique.
   b. $M = \sum M_i$ and $M_i \cap \sum_{j \neq i} M_j = \{0\}$.
   c. The homomorphism $\oplus M_i \to M$, defined by $\underline{m} \to \sum m_j$, is 1:1.

6. a. Let $A = \begin{bmatrix} 0 & y & x \\ y & x & 0 \end{bmatrix} \in Mat_{2\times 3}(\mathbb{k}[x,y])$. Consider A as a homomorphism $R^3 \to R^2$. (Dis)Prove:
   i. $Im(A) \supset (x,y) \cdot R^2$. ii. $Im(A) \supset (x,y)^2 \cdot R^2$. iii. The module $ker(A)$ is cyclic.
   Consider $coker(A)$ as a $\mathbb{k}$-vector space and compute $dim_{\mathbb{k}} coker(A)$.
   b. The annihilator of a module is $ann(M) := \{r \in R | \ r \cdot M = \{0\}\}$. Prove:
   i. $ann(M) \subseteq R$ is an ideal. ii. If $M \supseteq N$ then $ann(M) \subseteq ann(N)$ and $ann(N/M) \supseteq ann(N)$.
   Give examples with $ann(M) = ann(N/M)$, $ann(M) \subsetneq ann(N/M)$, $ann(M) \supsetneq ann(N/M)$.
   c. Prove: M is (naturally) a module over $R/ann(M)$. In particular:
   i. $\oplus \mathbb{Z}/n_i\mathbb{Z}$ is a module over $\mathbb{Z}/lcm\{n_i\}\mathbb{Z}$. ii. For $R^n \xrightarrow{A} R^n$ $coker[A]$ is a module over $R/(det(A))$.
   d. Prove: $ann(M)$ is the largest among the ideals satisfying: $M \in Mod - R/I$.
   e. Take any $0 \neq V_{\mathbb{k}}$ and some $T \in Hom_{\mathbb{k}}(V,V)$. Prove: V is a module over $\mathbb{k}[T]$ (the ring of polynomials in T). What is the annihilator of V?

7. Denote by $Hom_R(M,N)$ the set of all the homomorphisms. Verify: $Hom_R(M,N)$ is an R-module.
   a. Prove: $Hom_R(R,M) \cong M$.
   b. Compute: $Hom_{\mathbb{k}[x,y]}(\mathbb{k}[x,y]/(x^2-y^2), \mathbb{k}[x,y]/(x^2+y^2))$, here $char(\mathbb{k}) \neq 2$, $Hom_{\mathbb{Z}}(\mathbb{Z}/(6), \mathbb{Z}/(3))$, $Hom_{\mathbb{Z}}(\mathbb{Z}/(p), \mathbb{Z}/(q))$, here $p, q \in \mathbb{Z}$ are co-prime.
   c. Compute $Hom_R(R/I, R)$ for R a domain.