Introduction to Commutative Algebra

201.1.7071, Fall 2022, (D.Kerner)

Homework 4

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Questions to submit: 1.a. 1.b. 2.b. 2.d. 3.a. 4.b. 5.d. 6.c. (by e-mail)

Denote $0:_M I=\{v\in M|\ I\cdot v=0\}$. For two submodules $M,N\subseteq L$ denote $M:N=\{f\in R|\ f\cdot N\subseteq M\}$.

1. a. Take the submodule $M=Span_R[(2x-1,x,x^2+3),(x,x,x^2),(x+1,2x,2x^2-3)]\subset R^{\oplus 3}$, for $R=\mathbb{Q}[x]$.
   i. Prove: M is a free module. (Find its basis.)
   ii. Prove: ${}^{R^{\oplus 3}}/_{M}\cong {}^{R^{\oplus 2}}/_{R\cdot(x,3x-x^2+x^3)}$.
   b. Prove: the $\mathbb{Z}$-module ${}^{\mathbb{Q}}/_{\mathbb{Z}}$ is decomposable, not finitely-generated, and contains no (non-trivial) free submodules. Moreover, $ann({}^{\mathbb{Q}}/_{\mathbb{Z}})=0$ and $0:_{{}^{\mathbb{Q}}/_{\mathbb{Z}}}(n)\neq 0$ for each $n\in Z$.
   c. Consider an ideal $I\subset R$ as an R-module. When is this a free module? (A necessary and sufficient condition.)
   d. A module $M\in Mod(R)$ is called cyclic if it is generated by one element.
   Prove: If $0\neq M\in Mod-R$ is cyclic then $M\approx {}^{R}/_{ann(M)}$.

2. a. For submodules $M,N\subseteq L$ (dis)prove: i. $ann(M+N)=ann(M)\cap ann(N)$. ii. $M:N=ann\,{}^{M+N}/_{M}$.
   b. Given an ideal $I\subset R$ and a submodule $M\subset N$ verify: $I\cdot M\subset I\cdot N$. (Dis)Prove: $I\cdot M=M\cap(I\cdot N)$.
   c. Take the R-modules R^n,R^m (with their standard bases). Construct the natural isomorphism $Hom_R(R^n,R^m)\xrightarrow{\sim}Mat_{m\times n}(R)$. (In which sense it is natural?)
   d. Prove: $R^n\cong R^m$ (as R-modules) iff $m=n$. (There are several ways to do this.)

3. Fix $A\in Mat_{m\times n}(R)$. Define the left-right equivalence, $A\sim UAV^{-1}$, where $U\in GL(m,R)$, $V\in GL(n,R)$.
   a. For a local ring $(R,\mathfrak{m})$ prove: $A\sim \mathbb{1}_{r\times r}\oplus\tilde{A}$, where $\tilde{A}\in Mat_{(m-r)\times(n-r)}(\mathfrak{m})$.
   (What was this statement in $Lin.Alg_{\Bbbk}$?)
   b. Prove: if $A\sim\mathbb{1}\oplus\mathbb{O}$ then $coker[A]$ is a free R-module.

4. a. Prove: if M_1+M_2 and $M_1\cap M_2$ are f.g. then M_1,M_2 are f.g.
   b. Let R a domain. When is $Frac(R)$ a f.g. R-module? When is $R_{\mathfrak{p}}$ a f.g. R-module?
   c. Let $M\subset N$, where N is f.g. Is M necessarily f.g.? (Hint: check submodules of $\Bbbk[x,y]$, $C^\infty(\mathbb{R})$.)
   d. Take a vector space $V_{\Bbbk}$, not necessarily finitely generated. Prove: every linearly independent subset $S\subset V$ is contained in a basis of V. (In particular, V admits a basis.)

5. a. Establish the natural (basis-free) isomorphisms of vector spaces:
   $V^*\otimes_{\Bbbk}W^*\cong(V\otimes_{\Bbbk}W)^*\cong Hom(V,W^*)\cong Hom(W,V^*)$.
   b. Let R a domain, $M\in Mod-R$, and suppose $ann(M)\neq 0$. Compute $M\otimes_R Frac(R)$.
   c. Prove: $ann(M\otimes N)\supseteq ann(M)+ann(N)$. Does the equality hold? (${}^{\mathbb{Q}}/_{\mathbb{Z}}\otimes_{\mathbb{Z}}\mathbb{Q}$)
   d. Identify these rings (or express this in a short form) $\Bbbk[x]\otimes_{\Bbbk}\Bbbk[x]$, $\Bbbk[x]\otimes_{\Bbbk[x]}\Bbbk[x]$.

6. a. Given a homomorphism $R\to S$ prove: $S\otimes_R R[\underline{x}]\cong S[\underline{x}]$.
   b. For $\underline{x}=(x_1,\dots,x_n)$ fix some maps $\{R[\underline{x}]\ni x_i\to f_i(\underline{y})\in R[\underline{y}]\}_i$.
   Prove: this extends (uniquely!) to a homomorphism $R[\underline{x}]\to R[\underline{y}]$.
   c. The same question for the maps $\{R[[\underline{x}]]\ni x_i\to f_i(\underline{y})\in(\underline{y})\subset R[[\underline{y}]]\}_i$. (When proving uniqueness be careful with infinite sums.)
   d. Given a homomorphism of rings $R\to R'$ and $M\in mod-R$, $M'\in mod-R'$ (finitely generated).
   (Dis)Prove: i. $M\otimes_R R'$ is f.g. over R'. ii. M' is f.g. over R.
   iii. If M' is f.g. and R' is f.g. over R then M' is f.g. over R.