Introduction to Commutative Algebra

201.1.7071, Fall 2022, (D.Kerner)

Homework 5

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Questions to submit: 1.c. 1.e. 2.a. 2.c. 2.f. 3.b. 4.a. 4.b.iii. 4.c.i. (by e-mail)

We often consider a matrix $A \in Mat_{m\times n}(R)$ as the homomorphism of modules $R^{\oplus n} \to R^{\oplus m}$.

1. a. Given $M_2 \subset M_1 \subset M$ prove: $M/M_1 \cong {}^{M/M_2}\big/{}_{M_1/M_2}$.
   b. Prove: any homomorphism $\phi : M \to N$ induces the homomorphism $\bar{\phi} : M/I\cdot M \to N/I\cdot N$ of R/I-modules. When is $\bar{\phi} = 0$? When is $\bar{\phi}$ injective?
   c. Is a cylic module necessarily free?
   d. Prove: the ring $\mathbb{k}[[x]]$ is not countably-generated as a $\mathbb{k}[x]$-module. (Compare to q.5.a of hwk.2)
   e. Take $R = \mathbb{k}[t]$ and the homomorphisms: $R \to \mathbb{k}[\underline{x}], t \to f(\underline{x})$, and $R \to \mathbb{k}[\underline{y}], t \to g(\underline{y})$. Prove: $\mathbb{k}[\underline{x}]\otimes_R \mathbb{k}[\underline{y}] \cong \mathbb{k}[\underline{x},\underline{y}]/(f(\underline{x}) - g(\underline{y}))$.

2. Below R is a domain.
   a. Compute the rank of the $\mathbb{Z}$-module $\mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}/(3) \oplus \mathbb{Z}/(5)$.
   b. Consider an ideal $0 \neq I \subset R$ as an R-module. Compute $rank(I)$.
   c. i. Let $\mathbb{O} \neq A = [a, b, c] \in Mat_{1\times 3}(R)$. Compute the ranks of R-modules $Im(A)$ and $coker(A)$.
      ii. Compute the ranks of R-modules $Im(A^t)$ and $coker(A^t)$.
   d. Take $R \hookrightarrow Frac(R) =: \mathbb{K}$, accordingly $Mat_{m\times n}(R) \hookrightarrow Mat_{m\times n}(\mathbb{K})$, thus $A \rightsquigarrow A\otimes\mathbb{K} \in Mat_{m\times n}(\mathbb{K})$. (Dis)Prove: $rank(Im(A)) = rank(A\otimes\mathbb{K})$. (The latter rank is in the sense of $Lin.Alg._{\mathbb{K}}$)
   e. Express $rank(M \otimes N)$ via $rank(M)$, $rank(N)$.
   f. Prove: if M is generated by $n < \infty$ elements, then any $(n + 1)$-elements of M are linearly dependent. (We did this in the class)
      Deduce: $rank(M)$=the maximal number of linearly independent elements in M.

3. a. Go over all the detail of our proof of Cayley-Hamilton theorem. Why is the proof "$det(A\cdot\mathbb{1}-A) = 0$" wrong? How did we convert this into a valid proof?
   b. Here is another prooof. Write the details. Given $A \in Mat_{n\times n}(R)$ expand its characteristic polynomial, $p_A(t) := det[t\mathbb{1} - A] = \sum_{j=0}^{n} p_j t^j$. Observe: $p_n = 1$.
      Present $p_A(t)\cdot\mathbb{1} = (t\mathbb{1} - A)\cdot(t\mathbb{1} - A)^\vee$. Expand the adjugate, $(t\mathbb{1} - A)^\vee =: \sum_{j=0}^{n-1} B_j t^j$, for some matrices B_j over R. Write down explicit polynomials $p_j\cdot\mathbb{1} = pol_j(A, \{B_i\})$. Now expand $\sum_{j=0}^{n} A^j\cdot p_j\mathbb{1}$, and verify that this sum vanishes. (And all $\{B_i\}$ disappear.)

4. a. Let $R = \mathbb{k}[x, y]_{(x,y)}$ and fix some elements $p_3, q_3, r_3 \in (x, y)^3$. Prove: $(x, y)^2 = (x^2 - p_3, y^2 - q_3, xy - r_3) \subset R$.
   b. i. Consider $\mathbb{Q}$ as a $\mathbb{Z}$-module. Prove: $(n)\cdot\mathbb{Q} = \mathbb{Q}$.
      ii. Consider $\mathbb{k}[x, \frac{1}{x}]$ as a $\mathbb{k}[x]$-module. Prove: $(x)\cdot\mathbb{k}[x, \frac{1}{x}] = \mathbb{k}[x, \frac{1}{x}]$.
      iii. Consider $\mathfrak{m}^\infty$ as a $C^\infty(\mathbb{R}^1)$-module. (See q.6.e. of homework 0.) Prove: $\mathfrak{m}\cdot\mathfrak{m}^\infty = \mathfrak{m}^\infty$.
      iv. Does this contradict the Nakayama-lemma?
   c. i. Let $0 \neq M, N \in mod - R$ for a local ring R. Prove: $M \otimes_R N \neq 0$.
      ii. Give a counterexample when R is non-local or one of M, N is not f.g.