Introduction to Commutative Algebra

201.1.7071, Fall 2022, (D.Kerner)

Homework 6

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Questions to submit: 1.a. 1.c. 1.d. 2.a. 2.b.ii. 2.c. 3.a. 4.b. (by e-mail)

1. Take a local ring $(R, \mathfrak{m})$ and a morphism of f.g. modules $\phi : M \to N$.
   a. Suppose the morphism $\bar{\phi} : M/_{\mathfrak{m} \cdot M} \to N/_{\mathfrak{m} \cdot N}$ is surjective. Prove: ϕ is surjective. Suppose $\bar{\phi}$ is injective, is ϕ injective?
   b. Suppose a morphism $\phi : M \to M$ is surjective. Prove: ϕ is an isomorphism. Hint: fix some generators of M, take the presentation matrix $[\phi] \in Mat_{n\times n}(R)$. Prove: $ker[\phi] = 0 \subset R^n$. Conclude that $ker(\phi) = 0 \subset M$ (by the Cayley-Hamilton theorem).
   c. Take another morphism $\epsilon : M \to \mathfrak{m}\cdot N$. Prove: if ϕ is an isomorphism then $\phi+\epsilon$ is an isomorphism.
   d. A set of generators $\{v_i\}$ of M is called "a minimal generating set" if no v_i is an R-linear combination of the others. Prove: a finite set is a minimal generating set iff their images $\{\bar{v}_i\}$ form a basis of the vector space $M/_{\mathfrak{m} \cdot M}$.

2. a. Suppose the subset $V(I) \subset \Bbbk^n$ does not contain the origin. Identify the ideal $\Bbbk[\underline{x}]_{(\underline{x})} \cdot I \subseteq \Bbbk[\underline{x}]_{(\underline{x})}$.
   b. For $R = \Bbbk[x,y]/_{(xy(y-x))}$ identify the ring $S^{-1}R$ in the following cases. (What is the geometry?)
      i. S is generated by x. ii. S is generated by x, y. iii. S is generated by $R \setminus (y)$.
   c. Prove: every intermediate ring $\mathbb{Z} \subset R \subset \mathbb{Q}$ is obtained as the ring of fractions, $R = S^{-1}\mathbb{Z}$. (Hint: $\mathbb{Z}[\frac{3}{7}] = \mathbb{Z}[\frac{1}{7}]$.) Give examples that are not isomorphic to the ring $\mathbb{Z}[\frac{1}{d}]$.
   d. Suppose $R = R_1 \times R_2$. Present the projection $R \to R_1$ as the passage to the ring of fractions, $R \to S^{-1}R$, for some S.
   e. Verify: the relation used to define $S^{-1}R$ is indeed an equivalence relation, and $S^{-1}R$ is a (commutative, unital) ring.
   f. [Why we could not define $S^{-1}R$ just as $\{\frac{a}{s} | \ \frac{a_1}{s_1} \sim \frac{a_2}{s_2} \ if \ a_1s_2 = a_2s_1\}$?] Prove: if S contains zero divisors then "$\frac{a_1}{s_1} \sim \frac{a_2}{s_2} \ if \ a_1s_2 = a_2s_1$" is not an equivalence relation.
   g. Prove: the (canonical) map $R \to S^{-1}R$ is an isomorphism iff $S \subseteq R^{\times}$. When is $R \to S^{-1}R$ an embedding?

3. The homomorphism $\phi : R \to S^{-1}R$ induces the restriction and extension of scalars,
   $S^{-1}R \supset I \rightsquigarrow \phi^{-1}(I) \subset R$ and $R \supset I \rightsquigarrow \phi(I) \cdot S^{-1}R$.
   a. Let $I = (x^4 - y^5, y^6 - x^7) \subset \Bbbk[[x,y]]$ and $S = \langle x^3 + y^3 \rangle$. Prove: $I \cdot S^{-1}R = S^{-1}R$. (The geometry?)
   b. For any $I \subset S^{-1}R$ prove: $S^{-1}R \cdot \phi(\phi^{-1}(I)) = I \subset S^{-1}R$.
   c. For any $I \subset R$ prove: $\phi^{-1}(S^{-1}R \cdot \phi(I)) \supseteq I$. Give an example with inequality.
   d. Suppose $\mathfrak{p} \cap S = \varnothing$ for a prime $\mathfrak{p} \subset R$. Prove: $S^{-1}R \cdot \phi(\mathfrak{p}) \subset S^{-1}R$ is prime.

4. a. Suppose R is a domain and the multiplicative set S is generated by f. Prove: the restriction $S^{-1}R \supset I \rightsquigarrow I \cap R$ induces the embedding $\mathfrak{m}Spec(S^{-1}R) \hookrightarrow \mathfrak{m}Spec(R)$, whose image is $\mathfrak{m}Spec(R) \setminus V(f)$.
   b. Let R a domain and $0 \notin S \not\subseteq R^{\times}$ a multiplicative set. Prove: the R-module $R[S^{-1}]$ is not f.g.
   c. Given $M \in Mod - R$ define $S^{-1}M := M \otimes_R S^{-1}R$. Write the definition via the equivalence relation, $S^{-1}M = M \times S/_{(\dots)}$. Verify that this is the same module.
   d. Compute $S^{-1}M$ for $M = \oplus \mathbb{Z}/_{(n_i)} \in Mod - \mathbb{Z}$ and $S = \{a, b\}$ for some $0 \neq a, b \in \mathbb{Z}$.