# Introduction to Commutative Algebra

201.1.7071, Fall 2022, (D.Kerner)

## Homework 7

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Questions to submit: 1.b. 1.e. 2.c. 2.f. 3.a. 3.c. 4.b. 4.c. 5.c. (by e-mail)

1. a. Suppose a multiplicative set $S \subset R$ is generated by a (non-nilpotent) zero divisor $f$. Take $I = 0 :_R (f) = \{a|\ a \cdot f = 0\} \subset R$. Prove: $S^{-1}R \cong {}^{R}\!/\!_{I}[\frac{1}{f}]$. (What is the geometry?)
   b. Let $\mathfrak{m}_1, \mathfrak{m}_2 \subset R$ be distinct maximal ideals. Consider ${}^{R}\!/\!_{\mathfrak{m}_1}$ as $R$-module. Identify the localized module $({}^{R}\!/\!_{\mathfrak{m}_1})_{\mathfrak{m}_2}$.
   c. Let $A \in Mat_{n\times n}(R)$ such that $det(A) \in R$ is a non-nilpotent element. Let the multiplicative set $S$ be generated by $det(A)$. Identify the modules $S^{-1} \cdot ker(A)$, $S^{-1} \cdot coker(A)$. (What is the geometry? Note: $det(A)$ can be a zero divisor.)
   d. Let $R_1 = {}^{\Bbbk[x,y]}\!/\!_{(xy(x-y))}$, $R_2 = {}^{\Bbbk[x,y]}\!/\!_{(y^2-x^4)}$. Give an example of multiplicative sets $0 \notin S_1 \subset R_1$, $0 \notin S_2 \subset R_2$ satisfying: $S_1^{-1}R_1 \cong S_2^{-1}R_2$, and this ring is non-local.
   e. Fix an ideal $I \subset R$ and a multiplicative set $0 \notin S \subset R$, with the image $0 \notin \bar{S} \subset {}^{R}\!/\!_{I}$. Establish the isomorphism of $S^{-1}R$-modules: $\bar{S}^{-1}\,{}^{R}\!/\!_{I} \cong {}^{S^{-1}R}\!/\!_{S^{-1}I}$. (What is the geometry?)

2. We have postulated the closed sets on $Spec(R)$ as $V(I)$.
   a. Prove: this defines a topology on $Spec(R)$.
   b. Suppose $f \in R$ is nilpotent. Identify $V(f)$.
   c. For any subset $X \subset Spec(R)$ prove: $\bar{X} = V(\cap_{\{\mathfrak{p}\}\in X}\mathfrak{p})$. In particular, $\{\mathfrak{m}\} = V(\mathfrak{m}) \subset Spec(R)$ and these are the only closed points.
   d. Let $\Bbbk$ be $\mathbb{R}$ or $\mathbb{C}$. Find the (Zariski) closure in $Spec(\Bbbk[x,y])$ of the following sets:
      i. $\{(x,y)|\ (y-1)(y-x+1)(x+y+1) = 0,\ xy(x+y) \neq 0\}$.   ii. $\{(x,y)|\ y = sin(x)\}$.
   e. Prove: if $I \subsetneq J$ then $V(I) \supseteq V(J)$. Give an example with $V(I) = V(J)$.
   f. Let $R = \Bbbk[x]_{(x)}$. "Analyze" $Spec(R)$. What are the points, closed sets, residue fields? Take the mutliplicative set $S$ generated by $x$. Identify $S^{-1}R$ and $Spec(S^{-1}R)$.

3. For any $f \in R$ define "the basic open set" as $\mathcal{U}_f := Spec(R) \setminus V(f)$.
   a. Describe the sets $\mathcal{U}_x, \mathcal{U}_y \subseteq Spec({}^{\Bbbk[x,y]}\!/\!_{(xy)})$.
   b. Verify (for finite intersections): $\cap\mathcal{U}_{f_i} = \mathcal{U}_{\prod f_i}$.
   c. Prove: the basic opens form a *base* of the Zariski topology. Namely: any open subset of $Spec(R)$ is a union of basic opens.

4. To a homomorphism of rings $\phi : R \to S$ we have associated the map of spaces, $\phi^* : Spec(S) \to Spec(R)$.
   a. Identify (geometrically) the maps corresponding to: $\Bbbk[x] \hookrightarrow \Bbbk[x,y]$; $R \to {}^{R}\!/\!_{I}$; $\Bbbk[x] \to {}^{\Bbbk[x,y]}\!/\!_{(f)}$.
   b. Prove: the map $\phi^*$ is continuous.
   c. Prove (for $S$-reduced): the closure of the image is $\overline{Im(\phi^*)} = V(ker(\phi)) \subseteq Spec(R)$.
   d. Suppose $S$ is a domain. What is the image of the generic point of $S$?

5. Fix a module $M \in Mod\text{-}R$. An element $v \in M$ is called torsion if $r \cdot v = 0$ for a non-zero-divisor $r \in R$. Denote the set of all the torsion elements by $Torsion(M)$.
   a. Let $R = \mathbb{Z}$, thus $M$ is an abelian group. Characterize the torsion elements.
   b. Verify: $Torsion(M)$ is a submodule, and if $M$ is free then $Torsion(M) = \{0\}$.
   c. Let $R = {}^{\Bbbk[x,y]}\!/\!_{(x\cdot y)^2}$, $N = {}^{\Bbbk[x,y]}\!/\!_{(x)^2}$ and $M = {}^{\Bbbk[x,y]}\!/\!_{(x^2,xy,y^2)}$. Compute $Torsion(N)$, $Torsion(M)$.
   d. Prove: ${}^{M}\!/\!_{Torsion(M)}$ is a torsion-free module, i.e. $Torsion({}^{M}\!/\!_{Torsion(M)}) = 0$.