Introduction to Commutative Algebra

201.1.7071, Fall 2022, (D.Kerner)

Homework 8

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Questions to submit: 1.b. 1.f. 2.b. 2.d. 3.b. 3.c. 4.c. 5.b. (by e-mail)

1. a. Give a ring-automorphism of $\mathbb{C}[x]$, which is not a $\mathbb{C}$-algebra automorphism.
   b. Take a finitely generated multiplicative set $0 \notin S \subset R$. Prove: $S^{-1}R$ is a f.g. R-algebra. (Compare to q.4.b. of hwk.6)
   c. Prove: the $\Bbbk[x]$-algebra $\Bbbk[[x]]$ is not countably generated. (Compare to q.1.d of hwk.5)
   d. Under which condition on $\Bbbk$ is the $\Bbbk[x]$-algebra $\Bbbk[x]_{(x)}$ countably-generated?
   e. Denote by $\mathbb{F}_2$ the field with two variables. Denote $\mathbb{F}_4 := \mathbb{F}_2[x]/_{(x^2+x+1)}$.
      i. Verify: $\mathbb{F}_4$ is a field. (Prove: any finite domain is a field.)
      ii. Identify the algebra $\mathbb{F}_4 \otimes_{\mathbb{F}_2} \mathbb{F}_4$. Is this a domain?
   f. Suppose R is a domain/UFD/PID. Prove: $S^{-1}R$ is a domain/UFD/PID.
   g. Starting from $\mathbb{Z}$ or $\Bbbk[x]$ give an uncountable number of non-isomorphic PID's.

2. Let R be a Noetherian ring.
   a. (Dis)Prove: i. $S^{-1}R$ is Noetherian for a multiplicative set $0 \notin S$. ii. $R/_I$ is Noetherian. iii. Any subring of R is Noetherian. iv. Any f.g. algebra over R is Noetherian.
   b. Prove: the ring $R[[\underline{x}]]$ is Noetherian. (Follow the proof of the Hilbert basis theorem. Be careful with infinite series.)
   c. Prove: any intersection of closed sets, $\cap_j V(I_j) \subseteq Spec(R)$, stabilizes.
   d. Prove: for any closed subset $V(I) \subseteq Spec(R)$ and any open cover $V(I) \subset \cup \mathcal{U}_i$ there exists a finite subcover.
   e. Show that c. and d. do not hold for $R = C^\infty(\mathbb{R}^1)$, $R = C^\omega(\mathbb{R}^1)$.

3. Let $0 \to L \to M \stackrel{\phi}{\to} N \to 0$ be a s.e.s. of modules.
   a. Prove: M is f.g. iff L, N are f.g. (For the direction $\Rrightarrow$ assume R is Noetherian.)
   b. (R-Noetherian) For two submodules $M_1, M_2 \subset M$ prove: M_1, M_2 are f.g. iff $M_1 + M_2$ and $M_1 \cap M_2$ are f.g.
   c. For $M_1, M_2 \subset M$ (dis)prove: if $M_1 \cap L = M_2 \cap L$ and $\phi(M_1) = \phi(M_2)$ then $M_1 = M_2$.
   d. (Dis)Prove: $0 \to Torsion(L) \to Torsion(M) \to Torsion(N) \to 0$ is an exact sequence. (See q.5. of hwk.7.)
   e. Take an exact sequence of $\Bbbk$-vector spaces: $0 \to V_1 \to \cdots \to V_n \to 0$. Prove: $\sum (-1)^i dim_{\Bbbk}(V_i) = 0$.
   f. Take an exact sequence of modules over a domain: $0 \to M_1 \to \cdots \to M_n \to 0$. Prove: $\sum (-1)^i rank(M_i) = 0$. (Hint: verify that the sequence $0 \to M_1 \otimes Frac(R) \to \cdots \to M_n \otimes Frac(R) \to 0$ is exact.)

4. a. Prove: any exact sequence of vector spaces splits.
   b. We have proved in the class half of the spliting criterion of exact sequences. Prove the other half.
   c. Give an example of s.e.s. $0 \to L \to M \to N \to 0$, where $M \cong L \oplus N$, but the sequence does not split.
   d. Prove: any f.g. module over a Noetherian ring is presentable as $R^n \stackrel{A}{\to} R^m \to M \to 0$. (Here A is called "a presentation matrix".)

5. a. Let $M \in Mod\text{-}R$ and $0 \notin S \subset R$ a multiplicative set. Prove: $M = S^{-1}M$ iff $M \in Mod\text{-}S^{-1}R$.
   b. Suppose an R-module M is not finitely generated. (Dis)Prove: $rank(M) > 0$.
   c. Given R-modules $M \subsetneq N$, (dis)prove: $rank(M) \lneq rank(N)$.
   d. Prove: $S^{-1}M = 0$ iff $S \cap ann(M) \neq \varnothing$.